Modeling knowledge dynamics in multi-agent systems based on informants

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Abstract

In this paper we model knowledge dynamics in agents’ belief bases in a collaborative multi-agent system. Four change operators are introduced: expansion, contraction, prioritized revision, and non-prioritized revision. For all of them, both constructive definitions and an axiomatic characterization by representation theorems are given. We formally justify minimal change, consistency maintenance, and non-prioritization principles. These operators are based on an epistemic model for multi-source belief revision in which a rational way to weigh the beliefs using a credibility order among agents is developed. The defined operators can be seen as skills added to the agents improving the collective reasoning of a multi-agent system.

1 Introduction and motivation

Belief revision (BR) is the process of changing beliefs to take into account a new piece of information, observation, or evidence. The AGM paradigm (Alchourrón et al., 1985) has been widely accepted as a standard framework for belief revision; usually, Individual Belief Revision (IBR) in a single agent environment is achieved satisfying or adapting AGM postulates.

Single agent systems have evolved into multi-agent systems (MAS), where multiple interacting agents can collaborate, negotiate, discuss, etc., in order to achieve their goals. In many multi-agent domains and applications, each agent has its own initial beliefs as well as beliefs acquired from other informant agents. Hence, an agent can receive information from other agents which is contradictory with its own current beliefs. Therefore, IBR needs to be extended to multi-agent environments.

This paper focuses on Multi-Source Belief Revision (MSBR); i.e., belief revision performed by a single agent that can obtain new beliefs from multiple informants. Therefore, one of the contributions of our approach is the definition of an epistemic model for MSBR that considers both beliefs and meta-information representing the credibility of the belief’s source. We investigate how the belief base of an agent can be rationally modified when the agent receives information from other agents that can have different degrees of credibility. Thus, our main contribution is the definition based on the AGM model of four different belief change operators which use the credibility of informant agents in order to decide prevailing information. These operators are defined through constructive models and representation theorems that provide a complete axiomatic characterization for the proposed formalism.

An analysis of Belief Revision in Multi-Agent Systems was developed in (Liu and Williams, 1999) where the hierarchy of Figure 1 was introduced. There, the distinction of Multi-Agent Belief
Revision (MABR), Multi-Source Belief Revision (MSBR) and Single agent Belief Revision (SBR) is clearly explained. In contrast to MSBR, MABR investigates the overall belief revision behavior of agent teams or a society in which, in order to pursue the mutual goal, the agents involved need to communicate, cooperate, coordinate, and negotiate with one another. A MABR system is a MAS whose mutual goal involves belief revision. Different formalisms have been presented to deal with MABR (Liu and Williams, 1999, 2001, Kfir-Dahav and Tennenholtz, 1996, Malheiro et al., 1994). Nevertheless, in MSBR, an individual belief revision process is carried out in a multi-agent environment where the new information may come from multiple sources that may be in conflict. The main focus of our work is this kind of belief revision, also studied by (Dragoni et al., 1994).

![Belief Revision Hierarchy](image)

**Figure 1** Belief Revision Hierarchy.

Consider, for instance, the following (simplified) scenario. An agent $A$ has three informants $A_1$, $A_2$ and $A_3$ where, for $A$, $A_3$ is the most credible and the other two are equally credible. Agent $A$ wants to use a resource $S$, but it believes that $S$ is not available. Then $A_1$ informs $A$ that $S$ is available and hence, $A$ revises its beliefs to take into account this new piece of information. If then $A_3$ informs $A$ that $S$ is not available, since $A_3$ is more credible than $A_2$, then $A$ should change its belief about $S$. Observe that new information could be rejected if the new belief is informed by a less credible agent.

In the literature, there are several studied prioritized methods (e.g., partial meet revision (Alchourrón et al., 1985) and kernel revision (Hansson, 1999)). In these methods, the new information has priority over the beliefs in the base of the receiver agent. However, as is mentioned in (Fermé and Hansson, 1999) and (Falappa et al., 2002), in some scenarios a prioritized method can be unrealistic. Thus, some models of belief revision have been developed allowing for two options: either the new information is fully accepted or completely rejected (Hansson, 1997, Makinson, 1997, Hansson et al., 2001, Konieczny et al., 2010). For instance, if information comes from different sources, and these sources are not equally credible, a non-prioritized method can be more adequate. In contrast, if an agent always acquires information from the same source, then a prioritized method can be used.

In this paper, based on kernel revision, we develop a complete change model for MSBR where both non-prioritized and prioritized belief revision are defined. First, we propose a formalism for knowledge representation in a MAS; and then, based on this formalism we define different change operators for MSBR, either to add beliefs (expansion), to withdraw beliefs (contraction), or to maintain consistency (revision). We will introduce a credibility order among agents and, based on this order, a comparison criterion among beliefs is defined. In the revision process, if inconsistency arises, the credibility order is used to decide which information prevails. The contraction operator is based on kernel contraction (Hansson, 1994) and also uses the credibility order to decide which information prevails. We show that the proposed non-prioritized revision operator satisfies the minimal change principle, and incoming information can be rejected when the agent has more credible beliefs that contradict the new information. In the literature, there are other approaches that also attach information to agents’ beliefs that represents its credibility: (Benferhat et al., 1993), (Dragoni et al., 1994), (Cantwell, 1998) and (Benferhat et al., 2002). However, our approach differs from them as we will explain in detail below.

Some preliminary works related to this paper have been reported in two workshop papers (Tamargo et al., 2008, 2009). However, here, we extend both in several ways. We will define
different operators which describe a complete change model based on informants: expansion, contraction, prioritized revision and non-prioritized revision. These operators are based on an epistemic model in which a rational way to weigh beliefs is developed. These operators can be seen as skills added to the agents which improve the collective reasoning of a MAS. For these operators we give both constructive definitions and an axiomatic characterization of them by representation theorems. We also formally justify minimal change and consistency principles.

Thus, a complete change model where both prioritized and non-prioritized belief revision are defined. The rest of this paper is structured as follows. Next, Section 2 introduces the epistemic model for MSBR. Section 3 develops the concept of plausibility used to decide which beliefs will be preserved or erased in change operators. Section 4 defines change operators based on informants: expansion, two kinds of contractions, and two kinds of revisions. Section 5 presents forwarding information among agents. Finally, in Section 6 conclusions are given and related works are commented. All proofs can be found the Section 7.

2 Epistemic model for MSBR

In this section, we introduce an epistemic model for Multi Source Belief Revision which is based on informants. Then, in the following sections, we will define change operators based on agents interactions to add beliefs (expansion), to withdraw beliefs (contraction) and to revise beliefs. Note that the AGM model (Alchourrón et al., 1985) represents epistemic states by means of belief sets, that is, sets of sentences closed under logical consequence. Other models use belief bases; i.e., arbitrary sets of sentences (Fuhrmann, 1991, Hansson, 1992). Our epistemic model is based on an adapted version of belief bases which have additional information.

We adopt a propositional language $\mathbb{L}$ with a complete set of boolean connectives, namely $\{\lnot, \land, \lor, \rightarrow, \leftrightarrow\}$. Also, we assume the existence of an operator $Cn$ that satisfies inclusion ($B \subseteq Cn(B)$), iteration ($Cn(B) = Cn(Cn(B))$), monotonicity (if $B \subseteq C$ then $Cn(B) \subseteq Cn(C)$), and compactness (if $\alpha \in Cn(B)$, then $\alpha \in Cn(B')$ for some finite subset $B' \subseteq B$) and includes the classical consequence operator. In general, we will write $\alpha \in Cn(B)$ as $B \vdash \alpha$.

When interacting, agents will incorporate the received information into their knowledge base in form of information objects. An information object will associate a sentence with an agent. For the identification of the individual agents we introduce a finite set of agent identifiers that is denoted as $\mathbb{A} = \{A_1, \ldots, A_n\}$.

**Definition 1 (Information object)** An information object is a tuple $I = (\alpha, A_i)$, where $\alpha$ is a sentence of a propositional language $\mathbb{L}$ and $A_i \in \mathbb{A}$.

Information objects are used to represent an agent’s belief base. Observe that the agent identifier of an information object $I$ can be used for representing the agent from which the information is received, or the agent that has generated the information. In Section 5, we will describe different criteria for forwarding information that will determine which agent identifier will be used.

**Definition 2 (Belief base)** Let $\mathbb{A} = \{A_1, \ldots, A_n\}$ be a set of agent identifiers. A belief base of an agent $A_i$ ($1 \leq i \leq n$) is a set $K_{A_i} = \{I_1, \ldots, I_k\}$ containing information objects $(\alpha, A_j)$ ($1 \leq j \leq n$) received from other agents ($j \neq i$) and proper beliefs ($j = i$).

**Example 1** Consider the set of agent identifiers $\mathbb{A} = \{A_1, A_2, A_3, A_4\}$ and the belief base of the agent $A_1$, $K_{A_1} = \{((\beta, A_1), (\alpha, A_2), (\alpha, A_3), (\alpha \rightarrow \beta, A_4), (\omega, A_3), (\omega \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_1), (\alpha \rightarrow \delta, A_2), (\delta \rightarrow \beta, A_1), (\gamma, A_3), (\gamma, A_4), (\gamma \rightarrow \epsilon, A_2)\}$. Observe that $K_{A_1}$ has two information objects with the sentence $\alpha$; however, each one has a different agent identifier.

The set $\mathcal{K} = 2^\mathbb{L} \times \mathbb{A}$ will represent all the belief bases. Next, two auxiliary functions are introduced in order to obtain the set of sentences or the set of agents that belong to a belief base $K \in \mathcal{K}$.
Definition 3 (Sentence function) The sentence function \( \text{Sen} (\text{Sen} : \mathcal{K} \to 2^{\mathcal{C}}) \) is a function such that for a given belief base \( K \in \mathcal{K} \), \( \text{Sen}(K) = \{ \alpha : (\alpha, A_i) \in K \} \).

In our proposal, each agent \( A \in \mathbb{A} \) will have a consistent belief base \( K_A \). A belief base \( K_A \) is consistent if \( Cn(\text{Sen}(K)) \) is consistent.

Definition 4 (Agent identifier function) The agent identifier function \( \text{Ag} (\text{Ag} : \mathcal{K} \to 2^{\mathbb{A}}) \) is a function such that for every base \( K \in \mathcal{K} \), \( \text{Ag}(K) = \{ A_i : (\alpha, A_i) \in K \} \).

Example 2 Consider \( K_{A_1} \) of Example 1. Then,

- \( \text{Sen}(K_{A_1}) = \{ \beta, \alpha, \alpha \rightarrow \beta, \omega, \omega \rightarrow \beta, \alpha \rightarrow \delta, \delta \rightarrow \beta, \gamma, \gamma \rightarrow \epsilon \} \).
- \( \text{Ag}(K_{A_1}) = \{ A_1, A_2, A_3, A_4 \} \).

The agent identifier of an information object can be used to evaluate the truthfulness of the received information. In our approach, an assessment function will be used for representing the credibility each agent assigns to other agents. For defining this assessment, we assume a set of credibility labels \( \mathcal{C} = \{ c_1, \ldots, c_6 \} \) (common to all agents) with an order \( \prec_c \) such that for all \( c_1, c_2, c_3 \in \mathcal{C} \); if \( c_1 \prec_c c_2 \) and \( c_2 \prec_c c_3 \) then \( c_1 \prec_c c_3 \); \( c_1 \prec_c c_2 \) or \( c_2 \prec_c c_1 \); \( c_1 \prec_c c_1 \) does not hold; and if \( c_1 \prec_c c_2 \), then \( c_2 \prec_c c_1 \) does not hold. That is, following (Hein, 2010), we assume an irreflexive total order (also known strict total order).

Definition 5 (Assessment) Let \( \mathbb{A} = \{ A_1, \ldots, A_n \} \) be a set of agent identifiers and \( \mathcal{C} = \{ c_1, \ldots, c_6 \} \) a set of credibility labels. An assessment \( c_{A_i} \) for the agent \( A_i \) is a function \( c_{A_i} : \mathbb{A} \to \mathcal{C} \) assigning a credibility value from \( \mathcal{C} \) to each agent \( A_j \in \mathbb{A} \).

The set of credibility labels is the same for all agents; however, each agent will have its own assessment and different agents may have different assessments. Thus, the assessment of an agent can be replaced in a modular way without changing its belief base and without affecting other agents assessments.

Example 3 Consider the set of agent identifiers \( \mathbb{A} = \{ A_1, A_2, A_3, A_4 \} \) and the set of credibility labels \( \mathcal{C} = \{ c_1, c_2, c_3, c_4, c_5, c_6 \} \), where \( c_1 \prec_c c_2 \prec_c c_3 \prec_c c_4 \prec_c c_5 \prec_c c_6 \). The agents of \( \mathbb{A} \) can have the following assessments:

\[
\begin{align*}
A_1 : c_{A_1} (A_1) & = c_1, c_{A_1} (A_2) = c_2, c_{A_1} (A_3) = c_3, \text{ and } c_{A_1} (A_4) = c_3. \\
A_2 : c_{A_2} (A_1) & = c_2, c_{A_2} (A_2) = c_2, c_{A_2} (A_3) = c_2, \text{ and } c_{A_2} (A_4) = c_2. \\
A_3 : c_{A_3} (A_1) & = c_4, c_{A_3} (A_2) = c_3, c_{A_3} (A_3) = c_2, \text{ and } c_{A_3} (A_4) = c_1. \\
A_4 : c_{A_4} (A_1) & = c_4, c_{A_4} (A_2) = c_3, c_{A_4} (A_3) = c_2, \text{ and } c_{A_4} (A_4) = c_1.
\end{align*}
\]

Observe that for agent \( A_2 \) all agents have the same credibility, for agent \( A_3 \) all agents have different credibility, and agents \( A_2 \) and \( A_4 \) have the same assessment.

Thus, based on its own assessment, each agent can have a credibility order over the set \( \mathbb{A} \).

Definition 6 (Credibility order among agents) A credibility order among agents for an agent \( A_1 \), denoted by \( \leq_{C_0} \), is a total order over \( \mathbb{A} \) where \( A_1 \leq_{C_0} A_2 \) means that according to \( A_1, A_2 \) is at least as credible as \( A_1 \), and holds if \( c_{A_1} (A_1) \prec_c c_{A_1} (A_2) \) or \( c_{A_1} (A_1) = c_{A_1} (A_2) \). The strict relation \( A_1 \prec_{C_0} A_2 \), denoting that \( A_2 \) is strictly more credible than \( A_1 \), is defined as \( A_1 \leq_{C_0} A_2 \) and \( A_2 \not\leq_{C_0} A_1 \). Moreover, \( A_1 \preceq_{C_0} A_2 \) means that \( A_1 \) is as credible as \( A_2 \), and it holds when \( A_1 \leq_{C_0} A_2 \) and \( A_2 \preceq_{C_0} A_1 \).

Since \( \prec_{C_0} \), is a total order over \( \mathbb{A} \) then for all \( A_1, A_2, A_3 \in \mathbb{A} \) it holds:

- Reflexive: \( A_1 \leq_{C_0} A_1 \).
- Totality or Completeness: \( A_1 \leq_{C_0} A_2 \) or \( A_2 \leq_{C_0} A_1 \).
- Transitivity: if \( A_1 \preceq_{C_0} A_2 \) and \( A_2 \preceq_{C_0} A_3 \), then \( A_1 \preceq_{C_0} A_3 \).
Antisymmetry: if $A_1 \leq A_2$, and $A_2 \leq A_1$, then $A_1 = A_2$.

Example 4 Consider the set of agent identifiers $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ and the set of credibility labels $\mathcal{C} = \{c_1, c_2, c_3\}$, where $c_1 \prec c_2 \prec c_3$. Suppose that according to the assessment of $A_1$, $c_1(A_1) = c_1$, $c_1(A_2) = c_2$, $c_1(A_3) = c_2$, and $c_1(A_4) = c_3$. Then, the credibility order, according to $A_1$, is: $A_1 \leq A_2$, $A_1 \leq A_3$, $A_1 \leq A_4$, $A_2 \leq A_3$, $A_3 \leq A_4$, $A_2 \leq A_4$, and $A_3 \leq A_4$. Hence, $A_1 \leq A_2 = A_3 \leq A_4$.

The information received by an agent can be contradictory with its current beliefs. For instance, consider again the belief base $(K_{A_1})$ of Example 1, where $\text{Sen}(K_{A_1}) \vdash \beta$ (observe that there are several derivations for $\beta$). Suppose now that the agent $A_1$ receives the information object $I = (\neg \beta, A_4)$. It is clear that adding $(\neg \beta, A_4)$ to $K_{A_1}$ will produce an inconsistent belief base. Therefore, the agent has to decide whether it rejects $(\neg \beta, A_4)$ or withdraws $\beta$. In our approach, the credibility order `$\leq A_1$' will be used to decide which information prevails. If the new incoming information prevails, then the agent has to withdraw $\beta$. To do that, an adapted version of Kernel contractions will be introduced where all the minimal subsets of $K_{A_1}$ that entail $\beta$ will be considered.

Kernel contractions were introduced in (Hansson, 1994) and they are based on a selection among the sentences that are relevant to derive the sentence to be retracted. Note that kernel contractions are a generalization of safe contractions proposed in (Alchourrón and Makinson, 1985). In order to perform a contraction, kernel contractions use incision functions which cut into the minimal subsets that entail the information to be given up. Therefore, we will adapt the definition of $\alpha$-kernel to our epistemic model which will be used below to define a comparison criterion among sentences (called plausibility) and to define incision functions.

Definition 7 ($\alpha$-kernel) Let $K \in \mathcal{K}$ and $\alpha \in \mathcal{L}$. Then $H$ is an $\alpha$-kernel of $K$ if and only if

1. $H \subseteq K$.
2. $\text{Sen}(H) \vdash \alpha$.
3. If $H' \subset H$, then $\text{Sen}(H') \nvdash \alpha$.

Note that an $\alpha$-kernel is a minimal set of tuples from $K$ that entails $\alpha$. The set of $\alpha$-kernels of $K$ is denoted $K_{\alpha}$ and is called kernel set (Hansson, 1994).

Example 5 Consider $K_{A_1}$ of Example 1.

$K_{A_1}^+ = \{H_a, H_b, H_c, H_d, H_e, H_f, H_g, H_h\}$ where

$H_a = \{(\beta, A_1)\}$
$H_e = \{(\alpha, A_2), (\alpha \rightarrow \delta, A_1), (\delta \rightarrow \beta, A_1)\}$
$H_b = \{(\alpha, A_2), (\alpha \rightarrow \beta, A_4)\}$
$H_f = \{(\alpha, A_3), (\alpha \rightarrow \delta, A_1), (\delta \rightarrow \beta, A_1)\}$
$H_c = \{(\alpha, A_3), (\alpha \rightarrow \beta, A_4)\}$
$H_g = \{(\alpha, A_2), (\alpha \rightarrow \delta, A_2), (\delta \rightarrow \beta, A_1)\}$
$H_d = \{(\omega, A_4), (\omega \rightarrow \beta, A_4)\}$
$H_h = \{(\alpha, A_3), (\alpha \rightarrow \delta, A_2), (\delta \rightarrow \beta, A_1)\}$

Observe that a belief base can contain the same sentence in two (or more) information objects with different agent identifiers. For instance, in Example 1 $\{(\alpha, A_2), (\alpha, A_3)\} \subseteq K_{A_1}$. As it will be explained in detail below, when an agent $A_i$ receives an information object $(\alpha, A_j)$ consistent with its current belief base (i.e., $\text{Sen}(K_{A_i}) \nvdash \neg \alpha$), then $(\alpha, A_j)$ is added to $K_{A_i}$ (expansion). Note that it may be the case that $\text{Sen}(K_{A_i}) \vdash \alpha$; however, $(\alpha, A_j)$ is also added to $K_{A_i}$ because the credibility of the associated agent can increase the plausibility of $\alpha$. From the information objects point of view, there is no redundancy due to the fact that each information object represents a different informant.

Thus, in a belief base the same sentence can be in several information objects (with different agent identifiers). Therefore, if the assessment of the agent is changed and the credibility of a particular agent is increased, then all the sentences associated to this agent automatically have more credibility. Nevertheless, it will be useful that given a belief base $K$, a compacted belief base $K'$ can be obtained: i.e., a base where there are no tuples with the same sentence and the more
credible associated agent remains. In order to make more efficient the construction of changes, we propose that kernel sets can be computed over compacted belief bases.

Next, a function that given a belief base returns a compacted one is introduced (Definition 10). This function needs to know which is the most credible associated agent with respect to a given sentence, and this is returned by the top agent function.

**Definition 8 (Top agent function)** The top agent function, \( \text{Top} : \mathcal{L} \times \mathcal{K} \to 2^\mathcal{K} \), is a function such that for a given belief base \( K_A \in \mathcal{K} \) and a given sentence \( \alpha \in \text{Sen}(K_A) \), \( \text{Top}(\alpha, K_A) = \{ \alpha_k : (\alpha, \alpha_k) \in K_A, \text{ and for all } (\alpha, \alpha_j) \in K_A, \alpha_j \leq_{\mathcal{C}_0} \alpha_k \} \).

We assume that there is a function (see Definition 9) that based on a given policy\(^1\) returns a single agent identifier from a set of agent identifiers to which the assessment assigns the same credibility. In order to make more efficient the construction of changes, \( \text{Top} \) is straightforwardly deduced.

**Definition 9 (Selection function)** The selection function of an agent \( A_i \), \( \mathcal{S}_{A_i} : 2^\mathcal{K} \to \mathcal{A} \), is a function such that for a given set of agent identifiers with equal credibility with respect to the assessment of \( A_i \), it returns a single agent identifier based on a given policy.

**Definition 10 (Compact belief base function)** The compact belief base function (\( \text{Compact} : \mathcal{K} \to \mathcal{K} \)) is a function such that for a given belief base \( K_A \in \mathcal{K} \):

\[
\text{Compact}(K_A) = \{ (\alpha, A_i) : (\alpha, A_i) \in K_A, \text{ and } A_i = \mathcal{S}_{A_i}(\text{Top}(\alpha, K_A)) \}
\]

In order to simplify the notation we use \( K^1_A \) instead of \( \text{Compact}(K_A) \).

**Example 6** Consider again the agent \( A_1 \) of Example 1, where

\[
K_{A_1} = \{ (\beta, A_1), (\alpha, A_2), (\alpha, A_3), (\alpha \to \beta, A_4), (\omega, A_3), (\omega \to \beta, A_4), (\alpha \to \delta, A_1), (\alpha \to \delta, A_2), (\delta \to \beta, A_1), (\gamma, A_3), (\gamma, A_4), (\gamma \to \epsilon, A_2) \},
\]

and consider the credibility order among agents according to \( A_1 \) from Example 4, \( A_1 <_{\mathcal{C}_0} A_2 =_{\mathcal{C}_0} A_3 <_{\mathcal{C}_0} A_4 \). Then,

- \( \text{Top}(\gamma, K_{A_1}) = \{ A_4 \} \) and \( \text{Top}(\alpha \to \delta, K_{A_1}) = \{ A_2 \} \).
- \( \text{Top}(\alpha, K_{A_1}) = \{ A_2, A_3 \} \).
- \( \mathcal{S}_{A_1}(\{ A_2, A_3 \}) = A_2 \) where the policy adopted is based on a lexicographical ordering among agent identifiers.
- The compact belief base is:

\[
K^1_{A_1} = \{ (\beta, A_1), (\alpha, A_2), (\alpha \to \beta, A_4), (\omega, A_3), (\omega \to \beta, A_4), \} \cup \\
\cup \{ (\alpha \to \delta, A_2), (\delta \to \beta, A_1), (\gamma, A_4), (\gamma \to \epsilon, A_2) \}
\]

- \( K^1_{A_1} \models \alpha = \{ H_a, H_b, H_d, H_g \} \) where

\[
H_a = \{ (\beta, A_1) \} \quad H_d = \{ (\omega, A_3), (\omega \to \beta, A_4) \}
\]

\[
H_b = \{ (\alpha, A_2), (\alpha \to \beta, A_4) \} \quad H_g = \{ ((\alpha, A_2), (\alpha \to \delta, A_2), (\delta \to \beta, A_1) \}
\]

It is important to note that from the Definitions 3, 7, 8 and 10, the following proposition is straightforwardly deduced.

**Proposition 1** Let \( K_A \in \mathcal{K} \), it holds that: \( K^1_A \subseteq K_A \), \( \text{Sen}(K^1_A) = \text{Sen}(K_A) \), and \( K^1_A \models \alpha \subseteq K^\mathcal{C}_0 A \).

Consider \( \alpha \in \mathcal{L} \) and \( \beta \in \text{Sen}(K_A) \) such that \( \beta \) is in \( m \) tuples of \( K_A \) (\( m > 1 \)). Then \( K^1_A \subseteq K_A \). If \( X \in K^1_A \models \alpha \) and \( \beta \in \text{Sen}(X) \), then \( K^\mathcal{C}_0 A \) will have at least \( m \) \( \alpha \)-kernels differing only in the agent identifier of the tuple in which \( \beta \) is in. Nevertheless, since \( K^1_A \) has only one tuple containing \( \beta \), then \( K^1_A \models \alpha \subseteq K^\mathcal{C}_0 A \). In the following section, we will prove that it is equivalent to compute the plausibility of a sentence either with \( K^1_A \) or with \( K_A \).

\(^1\) A policy can be seen as a design decision.
3 Sentences plausibility

As stated above, when an agent $A_i$ receives an information object that is inconsistent with its knowledge base (e.g., it receives $(\neg \beta, A_1)$ and it holds that $\text{Sen}(K_{A_1}) \models \beta$), then the credibility order among agents $\leq_{C_0}$ will be used to decide which sentence prevails. Note that a sentence can have more than one derivation from a given knowledge base. Therefore, a comparison order among sentences (called Plausibility) will be defined. That is, if $\alpha$ and $\beta$ are sentences, the notation $\alpha \preceq_{K_{A_1}} \beta$ will represent the following: for the agent $A_i$, $\beta$ is at least as plausible as $\alpha$ relative to its assessment $c_{A_i}$ and its belief base $K_{A_i}$. The plausibility of a sentence will be used to define revision and contraction operators.

The concept of plausibility is related to epistemic entrenchment (Gärdenfors and Makinson, 1988) although the epistemic entrenchment orders are structured in a very specific way, and we apply it on belief bases instead of belief sets. According to (Gärdenfors, 1992), “...some sentences in a belief system have a higher degree of epistemic entrenchment than others... The guiding idea for the construction is that when a belief set $K$ is revised or contracted, the sentences in $K$ that are given up are those having the lowest degrees of epistemic entrenchment”.

The following function characterizes all the sentences that can be entailed from a belief base.

**Definition 11 (Belief function)** The belief function, $\text{Bel} : \mathcal{K} \rightarrow 2^\mathcal{L}$, is a function such that for a given belief base $K \in \mathcal{K}$, $\text{Bel}(K) = \{\alpha : \alpha \in \mathcal{L} \text{ and } \text{Sen}(K) \vdash \alpha\}$.

Similar to Proposition 1, note that from the Definitions 8, 10 and 11, the following proposition is straightforwardly deduced.

**Proposition 2** Let $K_{A_1} \in \mathcal{K}$, it holds that $\text{Bel}(K_{A_1}^\uparrow) = \text{Bel}(K_{A_1})$.

In order to calculate the plausibility of a sentence $\beta$, all its proofs have to be analyzed. Since we adopt a cautious approach, from each $\beta$-kernel we will consider those tuples that have the agent identifiers that are less credible. Two auxiliary functions are introduced below:

**Definition 12 (Least credible sources function)** The least credible sources function, $\text{min} : \mathcal{K} \rightarrow 2^\mathcal{K}$, is a function such that for a given belief base $K_{A_i} \in \mathcal{K}$, $\text{min}(K_{A_i}) = \{(\alpha, A_k) : (\alpha, A_k) \in K_{A_i} \text{ and for all } (\delta, A_j) \in K_{A_i}, A_k \leq_{C_0} A_j\}$.

**Definition 13 (Most credible sources function)** The most credible sources function, $\text{max} : \mathcal{K} \rightarrow 2^\mathcal{K}$, is a function such that for a given belief base $K_{A_i} \in \mathcal{K}$, $\text{max}(K_{A_i}) = \{(\alpha, A_k) : (\alpha, A_k) \in K_{A_i} \text{ and for all } (\delta, A_j) \in K_{A_i}, A_j \leq_{C_0} A_k\}$.

**Example 7** Consider the set of agent identifiers $\mathcal{A} = \{A_1, A_2, A_3\}$ and the credibility order of agent $A_1: A_1 <_{C_0} A_2 <_{C_0} A_3$. Let $K_{A_1} = \{(\alpha, A_1), (\alpha, A_2), (\beta, A_1), (\gamma, A_1), (\alpha \rightarrow \gamma, A_3)\}$ be the belief base of $A_1$. Then,

- $\text{min}(K_{A_1}) = \{(\alpha, A_1), (\beta, A_1), (\gamma, A_1)\}$,
- $\text{max}(K_{A_1}) = \{(\alpha \rightarrow \gamma, A_3)\}$.

Next, based on the agent comparison criterion $\leq_{C_0}$ of each agent, we will define a comparison criterion among sentences of $\text{Bel}(K_{A_i})$. First, we introduce the function $\text{Pl}(\alpha, K_{A_i})$ that given a sentence $\alpha \in \text{Bel}(K_{A_i})$, it returns an agent identifier that represents the plausibility of $\alpha$ with respect to the assessment of agent $A_i$. Then, based on the function $\text{Pl}$, in Definition 15, a comparison criterion $\leq_{K_{A_i}}$ among sentences of $\text{Bel}(K_{A_i})$ is introduced.

**Definition 14 (Plausibility function)** The plausibility function, $\text{Pl} : \mathcal{L} \times \mathcal{K} \rightarrow \mathcal{A}$, is a function such that for a given belief base $K_{A_i} \in \mathcal{K}$ and a sentence $\alpha \in \text{Bel}(K_{A_i})$:

$$\text{Pl}(\alpha, K_{A_i}) = \text{Ag}(\max\{\min(X) \mid X \in K_{A_i}^\uparrow, \alpha\})$$
Observe that the function max can return more than one agent identifier, therefore Pl uses the selection function $S_{A_1}$ of Definition 9 that returns only one identifier. Note also that it may be the case in which $(\gamma, A_1) \in K_{A_1}$ and $Pl(\gamma, K_{A_1}) \neq A_1$. For instance, consider Example 7, there $Pl(\alpha, K_{A_1}) = A_2$, $Pl(\beta, K_{A_1}) = A_1$ and $Pl(\gamma, K_{A_1}) = A_2$.

**Definition 15 (Plausibility criterion)** Let $K_{A_1} \in K$ be the belief base of agent $A_1$ and let $\{\alpha, \beta\} \subseteq \text{Bel}(K_{A_1})$, then $\alpha \preceq_{K_{A_1}} \beta$ if and only if it holds that $Pl(\alpha, K_{A_1}) \preceq_{\text{Co}_{\text{S}}(A)} Pl(\beta, K_{A_1})$.

Thus, the notation $\alpha \preceq_{K_{A_1}} \beta$ will represent: “for the agent $A_1$, $\beta$ is at least as plausible as $\alpha$”. The strict relation $\alpha <_{K_{A_1}} \beta$, representing “$\beta$ is more plausible than $\alpha$”, is defined as “$\alpha \preceq_{K_{A_1}} \beta$ and $\beta \not<_{K_{A_1}} \alpha$”. Moreover, $\alpha \sim_{K_{A_1}} \beta$ means that $\alpha$ is as plausible as $\beta$, and it holds when $\alpha \preceq_{K_{A_1}} \beta$ and $\beta \preceq_{K_{A_1}} \alpha$. From the previous definition we can observe that the plausibility of the sentences inherits the properties of the credibility order among agents ("$\preceq_{K_{A_1}}$" is a total order on $L$). Furthermore, note that the relation ‘$\preceq_{K_{A_1}}$’ is only defined with respect to a given $K_{A_1}$ (different belief bases may be associated with different orderings of plausibility).

**Example 8** Consider a set $A = \{A_1, A_2, A_3\}$. Suppose that agent $A_2$ has the following belief base $K_{A_2} = \{(\alpha, A_1), (\beta, A_2), (\gamma, A_3)\}$ and according to $A_2$ the credibility order is $A_1 <_{\text{Co}_{\text{S}}(A)} A_2 <_{\text{Co}_{\text{S}}(A)} A_3$. Furthermore, suppose that agent $A_3$ has the following belief base $K_{A_3} = \{(\alpha, A_1), (\beta, A_3), (\gamma, A_2)\}$ and the same credibility order than $A_2$, $A_1 <_{\text{Co}_{\text{S}}(A)} A_2 <_{\text{Co}_{\text{S}}(A)} A_3$. Then, for both agents, $\beta$ is more plausible than $\alpha$ (i.e., $\alpha <_{K_{A_2}} \beta$ and $\alpha <_{K_{A_3}} \beta$). However, for $A_2$, $\gamma$ is more plausible than $\beta$ ($\beta <_{K_{A_2}} \gamma$) whereas for $A_3$, $\beta$ is more plausible than $\gamma$ ($\gamma <_{K_{A_3}} \beta$). In this example $A_2$ and $A_3$ have the same assessment and $\text{Sen}(K_{A_2}) = \text{Sen}(K_{A_3})$ but their beliefs have different associated agents. It is clear that two agents with the same belief base but different credibility orders produce different orderings of plausibility. For instance, consider that $K_{A_1} = K_{A_2}$ and $A_2 <_{\text{Co}_{\text{S}}(A)} A_1 <_{\text{Co}_{\text{S}}(A)} A_3$ then $\alpha <_{K_{A_2}} \beta$ but $\beta <_{K_{A_1}} \alpha$.

The following example shows how the plausibility of a sentence can be calculated from a kernel set obtained from a compacted belief base.

**Example 9** Consider again Example 1, where the belief base of $A_1$ is $K_{A_1} = \{(\beta, A_1), (\alpha, A_2), (\alpha, A_3), (\alpha \rightarrow \beta, A_4), (\omega, A_4), (\omega \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_1), (\alpha \rightarrow \delta, A_2), (\delta \rightarrow \beta, A_1), (\gamma, A_3), (\gamma, A_3), (\gamma \rightarrow \epsilon, A_2)\}$, and consider the credibility order among agents according to $A_1$ from Example 4, $A_1 <_{\text{Co}_{\text{S}}(A)} A_2 <_{\text{Co}_{\text{S}}(A)} A_2 <_{\text{Co}_{\text{S}}(A)} A_4$. Then, suppose that agent $A_1$ needs to calculate the plausibility of $\beta$. In order to do so, $A_1$ will do the following steps.

- Step 1. Find the minimal subsets that derive $\beta$ from the compacted belief base $K_{A_1} (K_{A_1} \upharpoonright \beta)$. From Example 6 we can see that: $K_{A_1} \upharpoonright \beta = \{H_a, H_b, H_d, H_g\}$.
- Step 2. Apply ‘min’ to each $\beta$-kernel $\in K_{A_1} \upharpoonright \beta$:
  - $\text{min}(H_a) = \{(\beta, A_1)\}$
  - $\text{min}(H_d) = \{(\omega, A_3)\}$
  - $\text{min}(H_a) = \{(\alpha, A_2)\}$
  - $\text{min}(H_g) = \{(\delta \rightarrow \beta, A_1)\}$
- Step 3. Apply ‘max’ to the union of all the sets found in step 2.
  - $\text{max}((\beta, A_1), (\alpha, A_2), (\omega, A_3), (\delta \rightarrow \beta, A_1)) = \{(\alpha, A_2), (\omega, A_3)\}$
- Step 4. Find from the tuples of the previous item, the set containing the agent identifiers: $Ag\{(\alpha, A_2), (\omega, A_3)\} = \{A_2, A_3\}$.
- Step 5. Find from the set of agent identifiers of the previous item, a single agent identifier based on a given policy. For instance, if the policy is the lexicographical ordering among agent identifiers, then $S_{A_1}(\{A_2, A_3\}) = A_2$.

Therefore, $Pl(\beta, K_{A_1}) = A_2$. 
Proposition 3 shows that given a belief base $K_{A_i}$, the plausibility of a sentence can be obtained from either $K_{A_i}$ or $K_{A_i}^\uparrow$. However, applying the computation to $K_{A_i}$ requires computing more kernels than with $K_{A_i}^\uparrow$.

**Proposition 3** Let $K_{A_i} \in \mathcal{K}$ and let $\alpha \in \text{Bel}(K_{A_i})$, then the plausibility of $\alpha$ in the belief base $K_{A_i}$ is equal to the plausibility of $\alpha$ in the compacted belief base $K_{A_i}^\uparrow$. That is, $\text{Pl}(\alpha, K_{A_i}) = S_{A_i}(\text{Ag}(\max(\bigcup_{X \in K_{A_i}^\uparrow} \alpha \min(X)))) = S_{A_i}(\text{Ag}(\max(\bigcup_{X \in K_{A_i}} \alpha \min(X))))$.

*Proof:* See Appendix in Section 7.

Since the belief base of an agent may contain the same sentence in several different tuples, it could be natural to preserve only “the most plausible derivation” of each sentence. However, in the following example it is shown that this criterion may be problematic.

**Example 10** Consider $\mathcal{A} = \{A_1, A_2, A_3\}$ where $A_3 \prec_{C_0} A_2 \prec_{C_0} A_1$. Let $K_{A_2} = \{(\beta \rightarrow \alpha, A_2)\}$ be the belief base of $A_2$. Suppose that $A_2$ incorporates $(\beta, A_2)$ to $K_{A_2}$. In this scenario there are two derivations for $\alpha$, and $\text{Pl}(\alpha, K_{A_2}) = A_2$. Note that the plausibility of $\alpha$ was increased and it is unnatural to withdraw sentences from $K_{A_2}$ in order to preserve just one derivation of $\alpha$.

As we have shown in the previous example, it is very restrictive to have each sentence supported by only one derivation. For this reason, belief bases may be non-compacted. Thus, the plausibility of a sentence will be determined only by the plausibility function. Another reason for this decision is that we achieve a more dynamic framework since the evaluation of the credibility of the agent identifiers is separated by use of the assessment function.

It is important to note that the assessment function may change in time realizing dynamic assessments. Hence, the credibility order among agents can be changed without changing the knowledge base.

4 Change operators based on informants

In this section, we will define a change theory for multi-agent system focusing on Multi-Source Belief Revision (MSBR). The most widely studied model for belief revision is AGM model (Alchourrón et al., 1985) which distinguishes three change operators: expansions, contractions and revisions. The AGM model represents epistemic states by means of belief sets, that is, sets of sentences closed under logical consequence. Nevertheless, as introduced above, our epistemic model uses belief bases; that is, arbitrary sets of sentences. Next, in Section 4.1 we will define an expansion operator based on our epistemic model. Then in Section 4.2, we will introduce two contraction operators. In Section 4.3, we will define a prioritized revision operator and finally, in Section 4.4 we will propose a non-prioritized revision operator.

4.1 Expansion operator based on informants

In this subsection we will define an expansion operator for our epistemic model. This is the most simple operator to characterize from the logical point of view because it consists only in the addition of new information objects.

**Definition 16 (Expansion using plausibility)** Let $K_{A_i} \in \mathcal{K}$ the belief base of an agent $A_i$ and $(\alpha, A_j)$ an information object. The operator “$+$”, called expansion using plausibility, is defined as follows:

$$K_{A_i} + (\alpha, A_j) = K_{A_i} \cup \{(\alpha, A_j)\}$$

In contrast to the expansion proposed in (Hansson, 1999), here we consider information objects instead of single sentences. Therefore, if $\alpha \in \text{Bel}(K_{A_i})$, then this operation could increase
the plausibility of $\alpha$. This operation, as Hansson’s expansion, does not guarantee a consistent epistemic state.

Let $K_{A_i}, K_{A_j} \in \mathcal{K}$ two belief bases and let $A_i, A_j, A_k \in \mathcal{A}$. The expansion operator will be represented by “$+$”. We propose the following postulates for expansion using plausibility operator.

(EP-1) **Success:** $(\alpha, A_j) \in K_{A_i} + (\alpha, A_j)$.

The first postulate establishes that the expansion should be successful; i.e., the result of expanding a belief base $K_{A_i}$ by an information object $(\alpha, A_j)$ should be a new belief base that contains $(\alpha, A_j)$.

(EP-2) **Inclusion:** $K_{A_i} \subseteq K_{A_i} + (\alpha, A_j)$.

Obtaining information is a very expensive process, thus avoiding the unnecessary loss of information is wished in any change operator. Since $K_{A_i} + (\alpha, A_j)$ follows from adding an information object to $K_{A_i}$ without withdrawing any belief, it is natural to think that $K_{A_i}$ does not contain beliefs that do not belong to $K_{A_i} + (\alpha, A_j)$.

(EP-3) **Vacuity:** If $(\alpha, A_j) \in K_{A_i}$ then $K_{A_i} + (\alpha, A_j) = K_{A_i}$.

A particular case of expansion occurs when a belief base $K_{A_i}$ is expanded by an information object $(\alpha, A_j)$ which is in $K_{A_i}$. In this case, expanding $K_{A_i}$ by $(\alpha, A_j)$ does not generate any change in $K_{A_i}$.

(EP-4) **Monotonicity:** If $K_{A_j} \subseteq K_{A_i}$ then $K_{A_j} + (\alpha, A_k) \subseteq K_{A_i} + (\alpha, A_k)$.

Suppose that there are two belief bases and one of these is contained in the other. If both belief bases are expanded by the same belief then the inclusion relation between them should be preserved.

(EP-5) **Dynamic Plausibility:** If $\alpha \in \text{Bel}(K_{A_i})$ then $\text{Pl}(\alpha, K_{A_i}) \leq \text{Pl}(\alpha, K_{A_i} + (\alpha, A_j))$.

Suppose a belief base $K_{A_i}$ is expanded by an information object $(\alpha, A_j)$ where $\alpha \in \text{Bel}(K_{A_i})$. In this case, the result of expanding $K_{A_i}$ by $(\alpha, A_j)$ should not decrease the plausibility of $\alpha$. Thus, this operation could increase the plausibility of $\alpha$.

The postulates EP-1 . . . EP-5 characterize axiomatically our expansion operator. For all belief base $K$ and all information object $(\alpha, A_j)$, $K + (\alpha, A_j)$ is the smallest belief base which satisfies EP-1 . . . EP-5. Note that, EP-1 . . . EP-4 are defined in a similar way to those that define the expansion in AGM (Alchourrón et al., 1985), whereas the new postulate EP-5 considers the case that our belief base contains a belief with different associated agents.

### 4.2 Contraction operator based on informants

The contraction is a change operation which withdraws beliefs without adding anything. In practice, this situation occurs when an agent believes in $\alpha$ and perceives that $\neg \alpha$ is true. In this case, before adding $\neg \alpha$ it should be withdrawn $\alpha$. Even thought this operation gives rise to an adding of a new belief, can be decomposed in two operations: a contraction with respect to $\alpha$, and a subsequent expansion with respect to $\neg \alpha$. Below we introduce two contraction operators that are based on kernel contraction and adapted to our epistemic model.
4.2.1 Construction
In the belief base of an agent, several derivations for one sentence can exist. For instance, consider again the belief base \((K_{A_1})\) of Example 9. Suppose now agent \(A_1\) needs to retract \(\beta\) from its belief base. Since there are several derivations of \(\beta\), then it has to “cut” all of them. The credibility order will be used to decide which information prevails. For doing that, all the minimal subsets of \(K_{A_1}\) that entail \(\beta\) are obtained.

Kernel contractions are based on a selection among the sentences that are relevant to derive the sentence to be retracted. In order to perform a contraction, kernel contractions use incision functions which cut into the minimal subsets that entail the information to be given up. We will adapt this notion to our epistemic model. An incision function only selects information objects that can be relevant for \(\alpha\) and at least one element from each \(\alpha\)-kernel.

**Definition 17 (Incision function)** An incision function \(\sigma\) for \(K_{A_i} \in \mathcal{K}\) is a function such that for all \(\alpha\):

1. \(\sigma(K_{A_i}^\bot \alpha) \subseteq \bigcup(K_{A_i}^\bot \alpha)\), and
2. if \(\emptyset \neq X \in K_{A_i}^\bot \alpha\), then \(X \cap \sigma(K_{A_i}^\bot \alpha) \neq \emptyset\).

In the definition of an incision function in Hansson’s work it is not specified how the function selects the sentences that will be discarded of each \(\alpha\)-kernel. In our approach, this will be solved with the sentences plausibility that we have defined above. Thus, the incision function will select the least credible information objects of each \(\alpha\)-kernel.

**Definition 18 (Bottom incision function)** \(\sigma_1\) is a bottom incision function for \(K_{A_i}\) if \(\sigma_1\) is an incision function such that, \(\sigma_1(K_{A_i}^\bot \alpha) = \{\delta, A_k\} \in X \in K_{A_i}^\bot \alpha\) and for all \((\beta, A_j)\) \(\in X\) it holds that \(\delta \preceq_X \beta\)\(^2\).

**Example 11** Consider a set \(X = \{A_1, A_2, A_3\}\) where the credibility order according to \(A_2\) is \(A_1 \prec_{A_2} A_2 \prec_{A_2} A_3\). Suppose that the agent \(A_2\) has the following belief base \(K_{A_2} = \{(\alpha, A_1), (\beta, A_2), (\beta \rightarrow \alpha, A_1), (\beta \rightarrow \alpha, A_3), (\omega, A_1), (\omega \rightarrow \alpha, A_3), (\delta, A_1)\}\). Then, \(K_{A_i}^\bot \alpha = \{H_a, H_b, H_c, H_d\}\) where:

- \(H_a = \{(\alpha, A_3)\}\)
- \(H_b = \{(\beta, A_2), (\beta \rightarrow \alpha, A_3)\}\)
- \(H_c = \{(\beta, A_2), (\beta \rightarrow \alpha, A_1)\}\)
- \(H_d = \{(\omega, A_1), (\omega \rightarrow \alpha, A_3)\}\)

Then, the bottom incision function is:

\[
\sigma_1(K_{A_i}^\bot \alpha) = \{(\alpha, A_3), (\beta \rightarrow \alpha, A_1), (\beta, A_2), (\omega, A_1)\}
\]

Now that we have given the necessary background, two contraction operators will be defined. One of these operators (Definition 19) takes into consideration the whole belief base, and the other (Definition 20) considers its associated compacted belief base when an agent wants to apply the contraction operator.

**Definition 19 (Contraction using plausibility)** Let \(K_{A_i} \in \mathcal{K}\), \(\alpha \in \mathcal{L}\) and let \(\sigma_1\) be a bottom incision function for \(K_{A_i}\). The operator “\(\odot_{\sigma_1}\)”, called contraction using plausibility, is defined as follows:

\[
K_{A_i} \odot_{\sigma_1} \alpha = K_{A_i} \setminus \sigma_1(K_{A_i}^\bot \alpha)
\]

Note that when an agent wishes to contract its belief base for a sentence, it applies the contraction operator over the sentence and not over an information object. Furthermore, note that, it makes sense to have a version of contraction where the object to be contracted is a determined tuple (and not a sentence); however, we consider that it is not necessary for the aim of this article.

\(^2\)We assume that, given a relation \(\preceq_{K_{A_i}}\) on \(\mathcal{L} \times \mathcal{L}\), it is possible to define a relation \(\preceq_X\) on every \(X \subseteq K_{A_i}\).
Example 12  Consider the set $A = \{A_1, A_2, A_3, A_4\}$ where the credibility order according to $A_1$ is $A_1 < A_2 < A_3 < A_4$. Suppose that the agent $A_1$ has the following belief base $K_{A_1} = [(\beta, A_1), (\alpha, A_2), (\alpha \rightarrow \beta, A_2), (\alpha \rightarrow \beta, A_4), (\omega \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_2), (\delta \rightarrow \beta, A_1), (\gamma, A_3), (\beta, A_1), (\beta, A_4)]$. Then, suppose $A_1$ wants to contract by $\beta$ using $\sigma_{\leq A_1}$.

- **Step 1.** Find the minimal subsets that derive $\beta$ from $K_{A_1}$.
  
  $K_{A_1}^\downarrow \beta = \{H_a, H_b, H_c, H_d, H_f, H_g, H_h\}$ where
  
  $H_a = \{(\beta, A_1)\}$
  $H_b = \{(\alpha, A_2), (\alpha \rightarrow \beta, A_2)\}$
  $H_c = \{(\alpha, A_3), (\alpha \rightarrow \beta, A_2)\}$
  $H_d = \{(\alpha, A_2), (\alpha \rightarrow \beta, A_1)\}$
  $H_e = \{(\alpha, A_3), (\alpha \rightarrow \beta, A_1)\}$
  $H_f = \{(\beta, A_1), (\beta, A_4)\}$
  $H_g = \{(\beta, A_4)\}$
  $H_h = \{(\beta, A_1), (\beta, A_2)\}$

  - **Step 2.** Apply the bottom incision function $\sigma_{\leq A_1}$ to $K_{A_1}^\downarrow \beta$ to find the set containing the least credible information objects from each $\beta$-kernel.
  
  $\sigma_{\leq A_1}(K_{A_1}^\downarrow \beta) = \{(\beta, A_1), (\alpha, A_2), (\alpha \rightarrow \beta, A_2), (\alpha, A_3), (\omega, A_1)\} \cup \{(\delta \rightarrow \beta, A_1), (\gamma, A_3), (\beta, A_4)\}$

  - **Step 3.** $K_{A_1} \ominus_{\sigma_{\leq A_1}} \beta = K_{A_1} \setminus \sigma_{\leq A_1}(K_{A_1}^\downarrow \beta)$.
  
  $K_{A_1} \ominus_{\sigma_{\leq A_1}} \beta = \{(\alpha \rightarrow \beta, A_4), (\omega \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_2), (\beta \rightarrow \beta, A_1), (\gamma, A_3), (\beta, A_4)\}$.

  Note that, in Example 12 there are kernels that differ only in the associated agent identifier. This occurs when a base has the same sentence in several information objects. Since the incision function selects the least credible information objects of each $\alpha$-kernel, then as more information object containing the same belief are in $K_{A_1}$, more information objects will be selected by the bottom incision function. As a consequence, in some cases, this operator withdraws several information objects. In contrast, if the agent considers a compacted belief base when it applies the contraction operator, there will be less information objects selected by the incision function.

Definition 20 (Optimal contraction using plausibility)  Let $K_{A_1} \subseteq \mathcal{K}$, $\alpha \in \mathcal{L}$ and let $\sigma_{\downarrow}$ be a bottom incision function for $K_{A_1}$. The operator $\sigma_{\leq A_1}$, called optimal contraction using plausibility, is defined as follows:

$K_{A_1} \ominus \sigma_{\leq A_1} \alpha = K_{A_1} \setminus X$

where: $X = \{(\omega, A_j) : \omega \in \text{Sen}(\sigma_{\downarrow}(K_{A_1}^\downarrow \omega \alpha)) \text{ and } (\omega, A_j) \in K_{A_1}\}$.

Example 13  Consider $K_{A_1}$ and $\sigma_{\leq A_1}$ of Example 12. Then, suppose $A_1$ wants to contract by $\beta$ using $\sigma_{\leq A_1}$.

- **Step 1.** Find the minimal subsets that derive $\beta$ from a compacted belief base $K_{A_1}$.
  
  $K_{A_1}^\downarrow = \{(\beta, A_1), (\alpha, A_2), (\alpha \rightarrow \beta, A_4), (\omega, A_1), (\omega \rightarrow \beta, A_4)\} \cup \{(\alpha \rightarrow \delta, A_2), (\delta \rightarrow \beta, A_1), (\gamma, A_3), (\beta, A_4)\}$

  Note that, the policy used by the selection function (Definition 9), when we apply the compact belief base function (Definition 10), is based on a lexicographical ordering among agent identifiers.

  $K_{A_1}^\downarrow \ominus \beta = \{H_a, H_b, H_c, H_d, H_f\}$ where
  
  $H_a = \{(\beta, A_1)\}$
  $H_b = \{(\alpha, A_2), (\alpha \rightarrow \beta, A_2)\}$
  $H_c = \{(\alpha, A_3), (\alpha \rightarrow \beta, A_2)\}$
  $H_d = \{(\alpha, A_2), (\alpha \rightarrow \beta, A_1)\}$
  $H_f = \{(\omega, A_1), (\omega \rightarrow \beta, A_1)\}$
Step 2. Apply the bottom incision function “$\sigma_1^\downarrow$” to $K_A^\downarrow \beta$.

$\sigma_1(K_A^\downarrow \beta) = \{(\beta, A_1), (\alpha, A_2), (\omega, A_1), (\delta \rightarrow \beta, A_1), (\gamma, A_3), (\gamma \rightarrow \epsilon, A_2)\}$.

Step 3. $K_{A_1} - \sigma_1 \beta = \{(\alpha \rightarrow \beta, A_2), (\alpha \rightarrow \beta, A_4), (\omega \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_2), (\epsilon \rightarrow \beta, A_2), (\epsilon \rightarrow \beta, A_4)\}$.

Note that, since the belief base may be non-compacted, in step 3 of Example 13, all those tuples whose beliefs were selected by the bottom incision function without regarding the respective associated agents are discarded from $K_{A_1}$. Besides, observe that in the last two examples the contracted belief bases have the same beliefs (see Proposition 4). However, in the latest example, the belief base contains more information objects than in the previous one (see Proposition 5).

Then, this operator does not lose the associated agents of the belief remaining after the contraction. Consequently, this type of contraction is more conservative.

**Proposition 4** Let $K_{A_i} \in \mathcal{K}$, $\alpha \in \mathcal{L}$, “$\otimes_{\sigma_1}$” be a contraction using plausibility operator and “$\neg_{\sigma_1}$” an optimal contraction using plausibility operator, then

$$\text{Sen}(K_{A_i} \otimes_{\sigma_1} \alpha) = \text{Sen}(K_{A_i} -_{\sigma_1} \alpha)$$

*Proof:* See Appendix in Section 7.

**Proposition 5** Let $K_{A_i} \in \mathcal{K}$, $\alpha \in \mathcal{L}$, “$\otimes_{\sigma_1}$” be a contraction using plausibility operator and “$\neg_{\sigma_1}$” an optimal contraction using plausibility operator, then

$$K_{A_i} \otimes_{\sigma_1} \alpha \subseteq K_{A_i} -_{\sigma_1} \alpha$$

*Proof:* See Appendix in Section 7.

4.2.2 Properties

Next we will give the rationality postulates for optimal contraction using plausibility operator, adapting some of the postulates given in (Hansson, 1999), considering the following principle.

**Minimal change.** As much old knowledge as possible should be retained in the revised/contracted knowledge. That is, we should give up beliefs only when forced to do so, and then we should discard as few of them as possible.

Let $A_i$, $A_j$, $A_k$, $A_p \in \mathcal{A}$ and let $K_{A_i} \in \mathcal{K}$ be a belief base. The contraction operator will be represented by “$\neg$”. We propose the following postulates for contraction.

**(CP-1) Success:** If $\alpha \notin \text{Cn}(\emptyset)$, then $\alpha \notin \text{Bel}(K_{A_i} - \alpha)$.

The first postulate establishes that the contraction should be successful; i.e., the result of contracting a belief base $K_{A_i}$ by a sentence $\alpha$ (that is not a tautology) should be a new belief base that does not imply $\alpha$.

**(CP-2) Inclusion:** $K_{A_i} - \alpha \subseteq K_{A_i}$.

Since $K_{A_i} - \alpha$ follows from withdrawing some beliefs from $K_{A_i}$ without adding any belief, it is natural to think that $K_{A_i} - \alpha$ does not contain beliefs that do not belong to $K_{A_i}$.

**(CP-3) Uniformity:** If for all $K' \subseteq K_{A_i}$, $\alpha \in \text{Bel}(K')$ if and only if $\beta \in \text{Bel}(K')$ then $K_{A_i} - \alpha = K_{A_i} - \beta$.

This property establishes that if two sentences $\alpha$ and $\beta$ are implied by exactly the same subsets of $K_{A_i}$, then the contraction of $K_{A_i}$ by $\alpha$ should be equal to the contraction of $K_{A_i}$ by $\beta$. 
Next, we propose a new postulate which is an adapted version of the postulate of core-retainment defined in (Hansson, 1994): “The beliefs that we give up in order to contract $K_A$ by $\alpha$ should all be such that they contributed to the fact that $K_A - \alpha$, implies $\alpha$. More precisely, for $\beta$ to be deleted in the process of forming $K_A - \alpha$ from $K_A$, there should be some order in which the elements of $K_A$ can be removed, such that the removal of $\beta$ is the crucial step by which $\alpha$ stops to be logically implied.” In our contraction operator, this order is based on the credibility order among agents.

(CP-4) Minimal Plausibility Change: If $(\beta, A_p) \in K_A$ and $(\beta, A_p) \notin K_A - \alpha$ then there is $K' \subseteq K_A$ where $\alpha \notin Bel(K')$ but there exists $(\beta, A_j) \in K_A$ such that:

- $\alpha \in Bel(K' \cup \{ (\beta, A_j) \})$,
- $p = j$ or $A_p \leq_{Co} A_j$, and
- for all $(\delta, A_k) \in K'$ such that $\alpha \notin Bel((K' \cup \{ (\beta, A_j) \}) \setminus \{ (\delta, A_k) \})$ it holds that $A_j \leq_{Co} A_k$.

The intuition behind (CP-4) is that, if $\beta$ is removed, $\delta$ is preserved and both are used in a derivation of $\alpha$, then $\beta$ is removed because is less credible than $\delta$. In order to remove $\beta$ from $K_A$, we should withdraw from $K_A$ all the information objects containing $\beta$.

**Theorem 1** Let $K_A \in \mathcal{K}$ and let “$\oslash \sigma$” be a contraction operator. “$\oslash \sigma$” is an optimal contraction using plausibility for $K_A$ if and only if it satisfies CP-1, ..., CP-4, i.e., it satisfies success, inclusion, uniformity and minimal plausibility change.

**Proof:** See Appendix in Section 7.

Note that, from CP-4 it is straightforwardly possible verify the following remark.

**Remark 1** The optimal contraction using plausibility operator follows the principle of minimal change.

### 4.3 Prioritized revision using plausibility

In many multi-agent domains and applications, each agent has usually its own initial beliefs as well as knowledge acquired from other agents. In this Section and in Section 4.4, we develop two different ways in which the belief base of an agent can be rationally modified when the agent receives information from other agents that can have different degree of credibility. In the literature, there are several studied prioritized methods (e.g., partial meet revision (Alchourrón et al., 1985) and kernel revision (Hansson, 1999)). In these methods, the new information has priority over the beliefs in the base of the receiver agent. Our approach is based on kernel revision and the epistemic model defined above. Thus, we focus on MSBR, where agents maintain the consistency of their belief bases.

#### 4.3.1 Construction

The revision operator is the most complex change operator. This type of change guarantees a consistent epistemic state. When a belief base $K_A \in \mathcal{K}$ is revised by an information object $(\alpha, A_j)$ we will have two tasks:

- to maintain the consistency of $K_A$. If $\alpha$ is inconsistent with $Bel(K_A)$, that is $\lnot \alpha \in Bel(K_A)$, a deeper analysis is required because it is necessary to erase some information objects from $K_A$.
- to add $(\alpha, A_j)$ to $K_A$. This is the most simple task to characterize from the logical point of view because it consists only in the addition of new information object. As showed above, if $\alpha \in Bel(K_A)$ then this operation could increase the plausibility of $\alpha$. 
The first task can be accomplished contracting by $\neg\alpha$. The second task can be accomplished expanding by $(\alpha, A_j)$. If a belief base does not imply $\neg\alpha$, then $(\alpha, A_j)$ can be added without loss of consistency. This composition is based on the Levi identity (Gärdenfors, 1981, Alchourrón et al., 1985), which proposes that a revision can be constructed out of two sub-operations: a contraction by $\neg\alpha$ and an expansion by $(\alpha, A_j)$.

**Definition 21 (Prioritized revision using plausibility)**

Let $K_{A_i} \in K$, let $(\alpha, A_j)$ be an information object and let $\sigma_1$ a bottom incision function for $K_{A_i}$. Let $\neg\sigma_1$. be the optimal contraction using plausibility operator and $+$ the expansion using plausibility operator. The operator “*$\sigma_1$” , called prioritized revision using plausibility, is defined as follows:

$$K_{A_i} *_{\sigma_1} (\alpha, A_j) = (K_{A_i} - \sigma_1) \cap (\alpha, A_j)$$

**Example 14** Consider the set $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$ where the credibility order according to $A_i$ is $A_1 \prec \text{Bel}_o \prec A_2 \prec \text{Bel}_o \prec A_4 \prec \text{Bel}_o \prec A_5$. Suppose that the agent $A_1$ has the following belief base $K_{A_1} = \{(\beta, A_1), (\alpha, A_2), (\alpha, A_3), (\alpha \rightarrow \beta, A_4), (\alpha \rightarrow \beta, A_5)\}$. Then, suppose $A_1$ wants to revise by $(\neg\beta, A_2)$ using “*$\sigma_1$” . Since $\beta \in \text{Bel}(K_{A_1})$ then it is necessary to contract $K_{A_1}$ by $\beta$ and then expand $K_{A_1}$ by $(\neg\beta, A_5)$. Thus, $K_{A_1} *_{\sigma_1} (\neg\beta, A_5) = \{(\alpha \rightarrow \beta, A_2), (\alpha \rightarrow \beta, A_4), (\omega \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_2), (\gamma, A_3), (\gamma \rightarrow \epsilon, A_2), (\epsilon \rightarrow \beta, A_2), (\epsilon \rightarrow \beta, A_3)\} \cup \{\neg\beta, A_5\}$.  

### 4.3.2 Properties

Next we will give the rationality postulates for prioritized revision using plausibility operator. We must introduce the following principle, similar to the principle proposed in (Dalal, 1988).

**Maintenance of consistency.** If a belief base $K$ and a belief $\alpha$ are both consistent, then $K$ revised by $\alpha$ is consistent.

Let $A_i, A_j, A_k, A_p, A_q \in \mathcal{A}$ and let $K_{A_i} \in K$ be a belief base. The prioritized revision operator will be represented by “*”.

(RP-1) **Success:** $(\alpha, A_j) \in K_{A_i} * (\alpha, A_j)$.

Since the revision operator defined here is considered prioritized; i.e., the new information has priority, the first postulate we give establishes that the revision should be successful. That is, the result of revising a belief base $K_{A_i}$ by an information object $(\alpha, A_j)$ should be a new belief base that contains $(\alpha, A_j)$.

(RP-2) **Inclusion:** $K_{A_i} * (\alpha, A_j) \subseteq K_{A_i} \cup \{(\alpha, A_j)\}$.

A particular case in the revision process occurs when a belief base $K_{A_i}$ is revised by $(\alpha, A_j)$ and $\neg\alpha \in \text{Bel}(K_{A_i})$. In this case, before adding $(\alpha, A_j)$, $\neg\alpha$ should be withdrawn from $K_{A_i}$. Hence, if $\neg\alpha \in \text{Bel}(K_{A_i})$ then the revision of $K_{A_i}$ by $(\alpha, A_j)$ is contained in the expansion of $K_{A_i}$ by $(\alpha, A_j)$. In contrast, if $\alpha \in \text{Bel}(K_{A_i})$ (i.e., $\alpha$ is consistent with $K_{A_i}$) then the revision operation is equivalent to an expansion operation.

(RP-3) **Consistency:** if $\alpha$ is consistent then $K_{A_i} * (\alpha, A_j)$ is consistent.

The main aim of the revision operator is to hold consistency in the belief base revised. However, there exist special cases in which this is not possible. If a belief base is revised by an information object containing a contradictory sentence, then the resultant belief base is inconsistent. Hence, the revision operator should preserve the consistency in the belief base if and only if an information object containing a contradictory sentence is not being added to the belief base.
Uniformity: If for all $K' \subseteq K_A$, $\{\alpha\} \cup \text{Sen}(K') \vdash \bot$ if and only if $\{\beta\} \cup \text{Sen}(K') \vdash \bot$ then $K_A \cap (K_A \ast (\alpha, A_j)) = K_A \cap (K_A \ast (\beta, A_k))$.

This postulate determines that if two beliefs $\alpha$ and $\beta$ are inconsistent with the same sub-bases of $K_A$, then $K_A$ revised by information objects containing those beliefs should preserve the same information objects from $K_A$.

Minimal Plausibility Change: If $(\beta, A_p) \in K_A$ and $(\beta, A_p) \not\in K_A \ast (\alpha, A_k)$ then there is $K' \subseteq K_A$ where $\neg \alpha \not\in \text{Bel}(K')$ but there exists $(\beta, A_j) \in K_A$ such that:

- $\neg \alpha \in \text{Bel}(K' \cup \{(\beta, A_j)\})$,
- $p = j$ or $A_p \leq_{C_{A_j}} A_j$, and
- for all $(\delta, A_q) \in K'$ such that $\neg \alpha \not\in \text{Bel}((K' \cup \{(\beta, A_j)\}) \setminus \{(\delta, A_q)\})$ it holds that $A_j \leq_{C_{A_q}} A_q$.

The intuition behind this postulate is similar to that of the minimal plausibility change postulate (CP-4) for contractions introduced above.

**Theorem 2** Let $K_A \in K$ and let $\ast_{\sigma_1}$ be a revision operator. $\ast_{\sigma_1}$ is a prioritized revision using plausibility for $K_A$ if and only if it satisfies RP-1, ..., RP-5, i.e., it satisfies success, inclusion, consistency, uniformity and minimal plausibility change.

**Proof:** See Appendix in Section 7.

Note that, from RP-3 and RP-5, it is straightforwardly possible verify the following remark.

**Remark 2** The prioritized revision using plausibility operator follows the principles of minimal change and maintenance of consistency.

**Proposition 6** If “+” satisfies EP-1,...,EP-5 and “−_{\sigma_1}” satisfies CP-1,...,CP-4 then “\ast_{\sigma_1}” satisfies RP-1,...,RP-5.

**Proof:** See Appendix in Section 7.

### 4.4 Non prioritized revision using plausibility

A prioritized revision operator is characterized by success postulate from which we may infer that $\alpha \in \text{Bel}(K_A \ast (\alpha, A_j))$. That is, the incoming information has priority over the beliefs in the base of the receiver agent. However, as is mentioned in (Fermé and Hansson, 1999), this is an unrealistic feature, since actual epistemic agents, when confronted with information that contradicts previous beliefs, often reject it. Several models of belief revision have been developed that allow for two options: either the new information is completely accepted or it is completely rejected (Hansson, 1997, Makinson, 1997). Below, a non-prioritized revision operator for our proposed epistemic model is introduced.

When an agent always acquires information from the same source, then a prioritized method can be used. Nevertheless, if information comes from different sources, and this sources are not equally credible, a non-prioritized method can be more adequate. This occurs in many multi-agent domains and applications. Thus, we focus on a non-prioritized belief revision operator that is based on the credibility ordering among agents. We propose a method for analyzing the information received; if inconsistency arises, the credibility order is used to decide which information prevails. Thus, we show that, with this new revision operator, incoming information can not be accepted when the receiver agent has more credible beliefs that contradict the new information.
4.4.1 Construction
When a belief base $K \in \mathcal{K}$ is revised by an information object $I = (\alpha, A_j)$ using a non-prioritized revision operator there are two cases:

- $\alpha$ is consistent with $\text{Bel}(K)$. In this case, the operator is equivalent to the prioritized version.
- $\alpha$ is inconsistent with $\text{Bel}(K)$, that is $\neg \alpha \in \text{Bel}(K)$. First, it is necessary to determine whether the sentence will be accepted; and then if the input is accepted, then the operator is equivalent to the prioritized version.

According to this, two options arise: completely accept all the input, or completely reject all the input. In the literature there are other operators which may partially accept the new information, for instance, Revision by a Set of Sentences defined on belief bases (Falappa et al., 2002) and Selective Revision defined on belief sets (Fermé and Hansson, 1999).

**Definition 22 (Non-prioritized revision using plausibility)** Let $K_A, \beta$ be a belief base in $\mathcal{K}$, $(\alpha, A_j)$ be an information object, and $\sigma_1$, a bottom incision function for $K_A$. Let $*_{\sigma_1}$ be the prioritized revision using plausibility operator and $+$ be the expansion using plausibility operator. The operator $\circ_{\sigma_1}$, called non-prioritized revision using plausibility, is defined as follows:

$$K_A \circ_{\sigma_1} (\alpha, A_j) = \begin{cases} K_A + (\alpha, A_j) & \text{if } \neg \alpha \notin \text{Bel}(K_A) \\ K_A & \text{if } \neg \alpha \in \text{Bel}(K_A) \text{ and } A_j <_{\mathcal{C}_0} \text{Pl}(\neg \alpha, K_A) \\ K_A *_{\sigma_1} (\alpha, A_j) & \text{if } \neg \alpha \in \text{Bel}(K_A) \text{ and } \text{Pl}(\neg \alpha, K_A) \leq A_j \end{cases}$$

Note that, if the incoming information is as credible as the beliefs which are possibly withdrawn, this operator prioritizes the input. That is, if an agent receives information from the same informant, it is natural that the more recent information will be accepted.

**Example 15** Consider $K_A$ and $\beta <_{\mathcal{C}_0}$ of Example 14. Then, suppose $A_1$ wants to revise by $(\neg \beta, A_3)$ using $\circ_{\sigma_1}$. Since $\text{Pl}(\beta, K_{A_1}) = A_2 <_{\mathcal{C}_0} A_5$ then $K_{A_1} *_{\sigma_1} (\neg \beta, A_3) = K_{A_1} *_{\sigma_1} (\neg \beta, A_3) = \{(\alpha \rightarrow \beta, A_2), (\alpha \rightarrow \beta, A_4), (\alpha \rightarrow \delta, A_2), (\alpha \rightarrow \beta, A_2), (\epsilon \rightarrow \beta, A_4)\} \cup \{(\neg \beta, A_5)\}$.

It is important to note that if the input of Example 15 is $(\neg \beta, A_1)$ rather than $(\neg \beta, A_3)$, then the revision will not have effect because $A_1 <_{\mathcal{C}_0} A_2$. Thus, this operator will never discard more plausible sentences than the input.

4.4.2 Properties: success postulate must be weakened
An operator defined following the Definition 22, in general, satisfies the same postulates that satisfies the prioritized version. However, we must introduce the following principle.

**Non-prioritization principle.** If a belief base is revised by an information object $(\alpha, A_j)$, then $\alpha$ will not be necessarily accepted in the revised belief base. A sentence $\alpha$ will be accepted in the revised belief base only when its informant $A_j$ is sufficiently plausible or credible.

Let $A_i, A_j$ and $A_4 \in \mathcal{A}$, let $K_A \in \mathcal{K}$ be a belief base and let “$*$” be a prioritized revision operator on $K_A$. The non-prioritized revision operator will be represented by “$\circ$”. In the general case, the non-prioritized revision operator will be equal to a prioritized revision operator. However, in some particular cases, “$\circ$” does not satisfies success and, therefore, we need weaker versions of this postulate.

(NRP-1) **Weak Success:** if $\neg \alpha \notin \text{Bel}(K)$ then $(\alpha, A_j) \in K_A \circ (\alpha, A_j)$. 

This postulate establishes that $\alpha$ is accepted in the revised belief base if $\neg \alpha$ is not derived in the original belief base.

(NRP-2) **Relative Success:** $K_{A_1} \circ (\alpha, A_j) = K_{A_1} \circ (\alpha, A_j) \in K_{A_1} \circ (\alpha, A_j)$.

This postulate, inspired in (Hansson et al., 2001), says that all or nothing is accepted. That is, $\alpha$ is accepted in the revised belief base or nothing changes.

Both weak success and relative success do not capture the intuitions behind the non-prioritization principle. Therefore, we propose the following postulate, called Conditional Success.

(NRP-3) **Conditional Success:** If $(\beta, A_k) \in K_{A_1}^1$ and $\beta \notin Sen(K_{A_1} \circ (\alpha, A_j))$ then $(\alpha, A_j) \in K_{A_1} \circ (\alpha, A_j)$ if and only if $A_k \subseteq_{C_{A_1}} A_j$.

This postulate establishes that $\alpha$ is accepted in the revised belief base when its informant is sufficiently plausible.

Following Definition 22 it is possible to show that let $K_{A_1} \in \mathcal{K}$, "$\circ_{\sigma_1}$" is a non-prioritized revision using plausibility for $K_{A_1}$ if and only if it satisfies uniformity, consistency, conditional success, inclusion and minimal plausibility change. Hence, we can straightforwardly note that the non-prioritized revision using plausibility operator follows the principles of minimal change, maintenance of consistency and non-prioritization.

### 4.5 Application example

Consider the following scenario. An agent $A_1$ wants to travel to a village in a mountain and knows from the Tourist Information Office ($A_t$) that if it rains ($\rho$), then the road to the village is not open ($\neg \omega$). Hence, $K_{A_1} = \{(\rho \rightarrow \neg \omega, A_t)\}$. Agent $A_1$ also knows that it can obtain information from other sources: an agent $A_c$ coming down from the village, an agent $A_g$ at the gas station, an agent $A_r$ at some restaurant, or the weather report on the radio ($A_w$). The credibility order of $A_1$ is $A_1 <_{A_1} A_c <_{A_1} A_r <_{A_1} A_g <_{A_1} A_t <_{A_1} A_w <_{A_1} A_c$, and $A_1$ uses the non-prioritized operator $\circ_{\sigma_1}$ introduced above to revise its beliefs.

Then, the agent $A_1$ obtains from $A_r$ the information object $I_1 = (\rho, A_r)$ and revises its belief base: $K_{A_1} \circ_{\sigma_1} (\rho, A_r) = \{(\rho \rightarrow \neg \omega, A_t), (\rho, A_r)\}$. Observe that $I_1$ is added to its belief base, and now $\neg \omega \in Bel(K_{A_1})$ (i.e., with this new information $A_1$ believes that the road is not open). Later, $A_1$ obtains from $A_c$ the information object $I_2 = (\neg \rho, A_c)$. Since $\rho \in Bel(K_{A_1})$ and $A_r <_{A_1} Pl(\rho, K_{A_1}) = A_g$, then $I_2$ is rejected and its belief base does not change. Agent $A_1$ then obtains $I_3 = (\rho, A_w)$ from the weather report and revises $K_{A_1}$ by $I_3$. Since $I_3$ is not contradictory with $K_{A_1}$, $I_3$ is added: $K_{A_1} \circ_{\sigma_1} (\rho, A_w) = \{(\rho \rightarrow \neg \omega, A_t), (\rho, A_g), (\rho, A_w)\}$. Observe that the plausibility of $\rho$ and $\neg \omega$ are both increased.

Finally, $A_1$ obtains $I_4 = (\omega, A_c)$ (the road is open) from an agent $A_c$ that is coming down from the village. Since this new information is contradictory with $A_1$ beliefs (because $\omega \in Bel(K_{A_1}))$ then the kernel set for $\omega$ is obtained: $K_{A_1} \downarrow \omega = \{(\rho \rightarrow \neg \omega, A_t), (\rho, A_w)\}$. Then, $(\rho \rightarrow \omega, A_t)$ is selected to withdraw it, and hence, $K_{A_1} \circ_{\sigma_1} (\omega, A_c) = \{(\omega, A_c), (\rho, A_g), (\rho, A_w)\}$.

### 5 Forwarding information

In the previous sections we have introduced a formalism for Multi-Source Belief Revision. Using that formalism, agents can acquire information objects from multiple sources and incorporate
them into their proper beliefs. Both prioritized and non-prioritized revision operators were introduced using the credibility order of each agent in order to decide which information prevails. Nevertheless, nothing was said about how an agent can forward information that is obtained from others. In the following, we assume that all agents use the epistemic model introduced above, have their own credibility order, and incorporate information objects through some of the revision operators defined above.

Although the contribution of this paper is focused on the formalism presented above, in this section we briefly comment different strategies for forwarding information to other agents. In particular, we study how to rationally choose meta-information to be sent as a label in the information objects. The choice of the agent identifier to be sent with the piece of information is crucial as it influences the decision of the receiver about whether to accept the transmitted information. Thus, it is in the interest of the sending agent, and in fact in the interest of the whole coalition of agents, to choose this meta-information carefully.

As stated above, when an agent sends information to another agent, it sends information objects. Consider the set of agent identifiers $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ where $A_1 <_{C_0} A_2 =_{C_0} A_3 <_{C_0} A_4$. Suppose that the belief base of the agent $A_1$ is $K_{A_1} = \{(\alpha, A_2), (\alpha, A_4), (\beta, A_3), (\beta \rightarrow \alpha, A_1)\}$. If $A_1$ wants to send $\alpha$ to $A_2$, it should send a tuple $I = (\alpha, \text{Agent})$, and it is clear that there are several choices for the identifier “Agent” of $I$.

In (Krümpelmann et al., 2009), we describe different criteria for forwarding information that determine which agent identifier is considered by the receiver at the moment of reasoning. That is to say, we analyze different alternatives which determine which agent identifier is sent in the information object. Some of those are: “Sender identifier criterion” which, as in (Dragoni et al., 1994), suggests sending the proper sender ($A_1$) in the information object; “Source identifier criterion” that proposes sending one of the identifiers stored with $\alpha$ in the sender’s base (e.g., $A_2$, $A_4$) that can be one of them arbitrarily or the more credible of them as is suggested by the “Combined criterion”. In (Krümpelmann et al., 2009) is shown that there are some examples in which these simple criteria can select an unappropriated identifier.

In this section, we show a more elaborated criterion that takes the plausibility of sentences obtained from agents credibility into consideration. This criterion calculates the plausibility of a sentence $\alpha$ based on all its proofs before being forwarded. This calculation should return an agent identifier which will be used as the agent identifier of $\alpha$. Thus, a forwarding criterion can be implemented by sending an information object $I = (\alpha, A_i)$ where $A_i$ is the agent identifier obtained using the plausibility function defined above; i.e., $A_i = Pl(\alpha, K_{A_i})$ where $A_i$ is the forwarder. For instance, if the agent $A_1$ of application example of Section 4.5 wants to send the sentence $\rho$, it will send the information object $(\rho, A_w)$.

**Example 16** Let us consider Example 9 again. If the agent $A_1$ wishes to send $\beta$ to agent $A_4$ then, according to the plausibility based criterion, $A_1$ will send the information object $(\beta, Pl((\beta, K_{A_1})))$ to $A_4$. That is, $A_1$ sends, based on its belief base $K_{A_1}$ and its credibility order $\leq_{C_0}$, $(\beta, A_2)$ to $A_4$.

An important decision we made is to forward an agent identifier with a sentence rather than a credibility label in order to give additional information to the beliefs. One reason for this decision is that we achieve a more dynamic framework since the evaluation of the credibility of the agent identifiers is separated by the use of assessment function. Note that the assessment function may change in time realizing dynamic assessments. Hence, the credibility order among agents can be changed without changing the knowledge base or the operator. That is, if the credibility order among agents changes, then the plausibility of all sentences will also change without having to modify the belief base of the agent. Another reason for this decision is that since each agent has its own assessment (as stated in Section 2), it is more suitable to send agent identifiers and then the receiver agent can evaluate the received belief based on the credibility it has according to its own assessment. This means that the sending agent expresses that it considers the information
it transmits as credible as it considers the agent identifier in the information object. Now it is up to the receiver to assess how credible it considers each agent from its perspective using its own assessment function. We believe that this represents an advanced way of communication in multi-agent systems.

6 Conclusion and related work

In this work, we have proposed a general framework to deal with the knowledge dynamics of a multi-agent system. We have introduced an epistemic model and a set of operators to change the belief base of each agent: expansion, contraction, prioritized revision and non-prioritized revision. We have defined a set of postulates for every operator, we have proved representation theorems for the more important changes (contractions and revisions), and we have shown some interesting principles for each of them. We have shown how these operators can be used for multi-source belief revision system weighing beliefs following a credibility order among agents, giving illustrative examples, and showing that these operators may improve the collective reasoning of a multi-agent system.

In the literature, different formalisms have been presented to deal with MABR (Liu and Williams, 1999, 2001, Kfir-Dahav and Tennenholtz, 1996, Malheiro et al., 1994) where the overall belief revision of agent teams is investigated. In contrast to these, we focused on MSBR which is one of the essential components of MABR. Here, the agents maintain the consistency of their belief bases. Two other approaches that cope with MSBR are (Dragoni et al., 1994) and (Cantwell, 1998). The epistemic model in these works is similar to the one we defined in Section 2; however, our theory change is different to theirs. That is, like us, both consider that the reliability of the source affects the credibility of incoming information, and this reliability is used for making decisions. Nevertheless, these two approaches differ from ours in several issues as is detailed bellow.

In (Dragoni et al., 1994, 1997), it is considered that agents detect and store in tables the nogoods, which are the minimally inconsistent subsets of their knowledge bases. A good is a subset of the knowledge base such that: it is not inconsistent (it is not a superset of a nogood), and if augmented with whatever else assumption in knowledge base, it becomes inconsistent. In contrast to our approach, they do not remove beliefs to avoid a contradiction, but quite more generally, they choose which is the new preferred good among them in knowledge base. In our model, we obtain the kernel sets to cut some sentences, thus we break the contradictions if it is necessary.

Like us, they propose to store additional information with each sentence. However, their tuples contain 5 elements: <Identifier, Sentence, OS, Source, Credibility>, where Origin Set (OS) records the assumption nodes upon which it really ultimately depends (as derived by the theorem prover). In contrast to them, in our model a tuple only store a sentence and an associated agent, but a tuple does not store the credibility. That is, in our model, the plausibility of a sentence is not explicitly stored with it, as it is in (Dragoni et al., 1994). Thus, when the plausibility of some sentence is needed, the plausibility function should be applied. As is shown in Example 17, given a sentence α, its plausibility depends on its proofs (α-kernels). Therefore, if one of the sentences of these proof changes, then the plausibility of α may change. Hence, if the credibility order is replaced, then the sentence plausibility may change without changing the belief base.

Example 17 Consider a set $A = \{A_1, A_2\}$ where the credibility order is $A_1 \leq_c A_2$. $K_{A_1} = \{(\alpha, A_1), (\alpha \rightarrow \beta, A_2)\}$ and $K_{A_2} = \{(\alpha, A_2)\}$. By Definition 14, $Pl(\beta, K_{A_1}) = A_1$. Now, suppose that $A_1$ receives from $A_2$ the belief $\alpha$. Now $K_{A_1} = \{(\alpha, A_1), (\alpha, A_2), (\alpha \rightarrow \beta, A_2)\}$ and $A_1$ has two derivations for $\beta$, hence $Pl(\beta, K_{A_1}) = A_2$. Observe that plausibility of $\beta$ is increased.

The communication policy in (Dragoni et al., 1994) is that agents do not communicate the sources of the assumptions, but they present themselves as completely responsible for the knowledge they are passing on; receiving agents consider the sending ones as the sources of
all the assumptions they are receiving from them. In Section 5, we showed a more elaborated
criterion which proposes to calculate the plausibility of a sentence based on all its proofs before
being forwarded. An important decision we made was to forward an agent identifier with a
sentence rather than a credibility label in order to give additional information to the beliefs. One
reason for this decision is that since each agent has its own assessment, it is more suitable to
send agent identifiers so in this way, the receiver agent can evaluate the belief received based on
the credibility it has according to its own assessment. Another reason is that we achieve a more
dynamic framework since the evaluation of the credibility of the agent identifiers is separated by
use of the assessment function.

In (Cantwell, 1998), a scenario (set of incoming information) presented by a source is treated
as a whole and not sentence by sentence, and therefore, it can be inconsistent. A relation of
trustworthiness is introduced over sets of sources and not between single sources. Besides, if two
sources give the same piece of information \( \alpha \), and a single agent gives \( \neg \alpha \), then \( \alpha \) will be preferred,
that is, the decision is based on majority. In his approach, the order in which the evidence is
considered does not seem to be important. However, in our work, the order in which beliefs are
considered is important: If an agent receives \( \alpha \) and then receives \( \neg \alpha \) and both have the same
plausibility, then \( \neg \alpha \) will be rejected.

Our work has some link with the idea of epistemic entrenchment (Gärdenfors and Makinson,
1988, Rott, 1992). Here the sentence plausibility is used in a similar way to epistemic entrenchment
to modify knowledge. However, there are some differences between them. For instance, in our work
the order among sentences is based on the informants, whereas in (Gärdenfors and Makinson,
1988) the order among sentences is implicitly defined over belief states represented by belief sets.

When we count on a multi-agent system that has only one agent, the new operator is very
drastic. In this scenario there is no order among agents. The same happens when all the agents
of a multi-agent system have equal credibility. In these cases the bottom incision function does
not have enough information to select sentences and it will erase all sentences in the \( \alpha \)-kernels.
This behavior is similar to full meet revision on belief bases (Hansson, 1999). Nevertheless, when
a multi-agent system has several agents with different credibility and it is necessary to represent
knowledge dynamics of the agents, plausibility seems to be a good criteria.

In (Benferhat et al., 1993), several methods to deal with inconsistency are investigated by
defining notions of consequence capable of inferring non trivial conclusions from an inconsistent
knowledge base. It is clear that the methods proposed here and in (Benferhat et al., 1993) follow
different attitudes when facing inconsistent knowledge. In (Benferhat et al., 1993) inconsistency-
tolerant consequence relations in layered knowledge bases are proposed, whereas here a revision
operator is defined.

It is important to note that the revision operator proposed here is similar to the revision
operator proposed in (Benferhat et al., 2002). However, these operators are built in a different
way. In (Benferhat et al., 2002), the epistemic state is represented by a possibility distribution
which is a mapping from the set of classical interpretations or worlds to the \([0,1]\) interval.
This distribution represents the degree of compatibility of the interpretations with the available
information and the revision is done over the possibility distribution. This revision modifies the
ranking of interpretations so as to give priority to the input information. The input must be
incorporated in the epistemic state; in other words, it takes priority over information in the
epistemic state. They discuss the revision with respect to uncertain information; the input is
of the form \((\phi, a)\), which means that the classical formula \( \phi \) should be believed to a degree of
certainty of exactly \( a \).

Both approaches differ in some interesting ways. A first difference occurs in the way they
handle the epistemic state. In (Benferhat et al., 2002), the authors use belief sets, whereas we use
belief bases. The use of belief bases makes the representation of the agent’s cognitive state more
natural and computationally tractable. That is, following (Hansson, 1999, page 24), we consider
that agents’ beliefs could be represented by a limited number of sentences that correspond to the
explicit beliefs of the agent. Another important difference, related to the intention of using the operator in a MAS environment, is the additional information added to each belief. Here, to decide whether to reject or accept a new belief, a comparison criterion among beliefs is defined. This characteristic is one of the motivations for using agent identifiers instead of representing the plausibility of a sentence as in (Benferhat et al., 2002). Moreover, here a total order among agents is necessary, but this assumption can be relaxed considering a partial order among agents.

As future work, we will try to extend the applications of the proposed framework in environments in which the credibility of agents changes, and therefore, the plausibility of beliefs (and the results of changes) can be dynamically modified. That is, since the revision process is based on the credibility order among agents, it is possible to define an operator to revise the credibility order. This will allow to represent changes over the credibility order.

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7 Appendix

Proposition 3. Let \( K_{A_i} \in \mathcal{K} \) and let \( \alpha \in Bel(K_{A_i}) \), then the plausibility of \( \alpha \) in the belief base \( K_{A_i} \) is equal to the plausibility of \( \alpha \) in the compacted belief base \( K_{A_i}^{1} \). That is, \( Pl(\alpha, K_{A_i}) = S_{A_i}(Ag(max(\bigcup_{X \in K_{A_i}^{1}} min(X)))) = S_{A_i}(Ag(max(\bigcup_{X \in K_{A_i}^{1}} min(X)))) \).

Proof Let \( \mathcal{A} = \{ A_1, \ldots, A_n \} \). If \( K_{A_i}^{1} = K_{A_i} \) then is trivially proved. If \( K_{A_i} \subseteq K_{A_i} \) then there exists some sentence \( \beta \) in \( Sen(K_{A_i}) \) such that \( \beta \) occurs in \( m \) tuples in \( K_{A_i} \) \((m \geq 2)\). Consider \( (\beta, A_i) \in X \) \((1 \leq i \leq n)\) for some \( X \in K_{A_i}^{1} \) \( \alpha \) then \( K_{A_i}^{1} \alpha \) will have \( m \) \( \beta \)-kernels \((X, Y_1, \ldots, Y_{m-1})\) such that they will differ only in the tuple containing \( \beta \). Suppose that \( (\beta, A_j^p) \in Y_p \) for all \( p \) \((1 \leq p \leq m - 1, j \neq i) \) and \( 1 \leq j \leq n) \). Next, we will prove that \( X \) will contain the only relevant tuples to compute the plausibility of \( \alpha \). There are three cases:

- If \( min(X) = (\beta, A_i) \), then we have that \( min(Y_p) = (\beta, A_i^p) \). By Definition 10, \( A_i^p \leq \alpha \) for all \( p \). Moreover \( X, Y_1, \ldots, Y_{m-1} \) differ only in the tuple in that is \( \beta \). Therefore, \( max((\beta, A_i), (\beta, A_i^1), \ldots, (\beta, A_i^{m-1})) = (\beta, A_i) \) \in X \). In case that \( A_i^p = A_i \), note that the selection function follows the same policy either in compact belief base function as plausibility function. Hence, the selection function returns the same agent identifier in both cases.

- If \( min(X) \neq (\beta, A_i) \) and \( min(Y_p) \neq (\beta, A_i^p) \), then \( min \) will return the same tuple in all the cases. Then \( X, Y_1, \ldots, Y_{m-1} \) differ only in the tuple containing \( \beta \).

- If \( min(X) \neq (\beta, A_i) \) (suppose that \( min(X) = (\omega, A_j) \)) and \( min(Y_p) = (\beta, A_i^p) \) for some \( p \), then \( (\omega, A_j) \in Y_p \), \( A_i^p \leq A_i \). Note that, if \( min(Y_p) \neq (\beta, A_i^p) \) then the previous case \( min(Y_p) = (\omega, A_j) \). Hence, \( max(min(X) \cup min(Y_1) \cup \ldots \cup min(Y_{m-1})) = (\omega, A_j) \in X \). In case that \( A_i^p = A_i \), note that the selection function follows the same policy either in compact belief base function as plausibility function. Hence, the selection function returns the same agent identifier in both cases.

Therefore, from the \( m \) \( \beta \)-kernels \((X, Y_1, \ldots, Y_{m-1})\) only \( X \) will contain the relevant tuples to calculate the plausibility of \( \alpha \). Then, \( Pl(\alpha, K_{A_i}) \) is equal to:

\[
S_{A_i}(Ag(max(\bigcup_{X \in K_{A_i}^{1}} min(X)))) = S_{A_i}(Ag(max(\bigcup_{X \in K_{A_i}^{1}} min(X))))
\]
Proposition 4. Let $K_A \in \mathcal{K}$, $\alpha \in \mathcal{L}$, “$\ominus_{\sigma_i}$” be a contraction using plausibility operator and “$-_{\sigma_i}$” an optimal contraction using plausibility operator, then

$$\text{Sen}(K_A \ominus_{\sigma_i} \alpha) = \text{Sen}(K_A -_{\sigma_i} \alpha)$$

Proof

$(\subseteq)$ Let $\beta \in \text{Sen}(K_A \ominus_{\sigma_i} \alpha)$. We should prove that $\beta \in \text{Sen}(K_A -_{\sigma_i} \alpha)$. Then, by Definition 3 there exists an information object $(\beta, A_j) \in K_A \ominus_{\sigma_i} \alpha$. It follows from Definition 19 that $(\beta, A_j) \in (K_A \setminus \sigma_1(K_A^\bot \alpha))$. Thus $(\beta, A_j) \in K_A$ and $(\beta, A_j) \not\in \sigma_1(K_A^\bot \alpha)$. Since by Proposition 1 $K_A^{\top -\bot} \alpha \subseteq K_A^\bot \alpha$ then, by Definition 18, $\sigma_1(K_A^{\top -\bot} \alpha) \subseteq \sigma_1(K_A^\bot \alpha)$ and $(\beta, A_j) \not\in \sigma_1(K_A^{\top -\bot} \alpha)$. Hence, by Definition 3, $\beta \not\in \text{Sen}(\sigma_1(K_A^{\top -\bot} \alpha))$ and, by Definition 20, $\beta \in \text{Sen}(K_A -_{\sigma_i} \alpha)$.

$(\supseteq)$ Let $\beta \in \text{Sen}(K_A -_{\sigma_i} \alpha)$, We should prove that $\beta \in \text{Sen}(K_A \ominus_{\sigma_i} \alpha)$. Then, by Definition 3, there exists an information object $(\beta, A_j) \in K_A -_{\sigma_i} \alpha$. It follows from Definition 20 that $(\beta, A_j) \in K_A \setminus X$ where $X = \{ \omega, A_k : \omega \in \text{Sen}(\sigma_1(K_A^\bot \alpha)) \}$ and $(\omega, A_k) \in K_A_j$. Thus $(\beta, A_j) \in K_A$ and $(\beta, A_j) \not\in X$. Then there exists $(\beta, A_j) \in K_A$ such that $(\beta, A_j) \not\in \sigma_1(K_A^\bot \alpha)$ with $A_j \subseteq A_p$. In case that $(\beta, A_p) \not\in U(K_A^\bot \alpha)$ then, by Definition 18, $(\delta, A_q) \not\in \sigma_1(K_A^\bot \alpha)$. Since $(\beta, A_p) \not\in \sigma_1(K_A^\bot \alpha)$, it follows from Definition 18 that there exists $(\delta, A_q) \in Y \in K_A^{\top -\bot} \alpha$ such that $A_q \subseteq A_p$. Thus $(\delta, A_q) \in K_A$ and, by Proposition 1, $(\delta, A_q) \in K_A$ then for all $Z \in K_A^\bot \alpha$ such that $\text{Sen}(Y) = \text{Sen}(Z)$, $(\beta, A_p) \not\in Z \cap \sigma_1(K_A^\bot \alpha)$. Hence, by Definition 19, $\beta \in \text{Sen}(K_A \ominus_{\sigma_i} \alpha)$.

Proposition 5. Let $K_A \in \mathcal{K}$, $\alpha \in \mathcal{L}$, “$\ominus_{\sigma_i}$” be a contraction using plausibility operator and “$-_{\sigma_i}$” an optimal contraction using plausibility operator, then

$$K_A \ominus_{\sigma_i} \alpha \subseteq K_A -_{\sigma_i} \alpha$$

Proof

Let $(\beta, A_j) \in K_A \ominus_{\sigma_i} \alpha$, we should prove that $(\beta, A_j) \in K_A -_{\sigma_i} \alpha$. It follows from Definition 19 that $(\beta, A_j) \in (K_A \setminus \sigma_1(K_A^\bot \alpha))$. Thus $(\beta, A_j) \in K_A$ and $(\beta, A_j) \not\in \sigma_1(K_A^\bot \alpha)$. Since by Proposition 1 $K_A^{\top -\bot} \alpha \subseteq K_A^\bot \alpha$ then, by Definition 18 $\sigma_1(K_A^{\top -\bot} \alpha) \subseteq \sigma_1(K_A^\bot \alpha)$. Thus $(\beta, A_j) \not\in \sigma_1(K_A^{\top -\bot} \alpha)$. Hence, by Definition 3, $\beta \not\in \text{Sen}(\sigma_1(K_A^{\top -\bot} \alpha))$. Then, by Definition 20, $(\beta, A_j) \in K_A -_{\sigma_i} \alpha$.

Next, we give a lemma used in the representation theorem of contraction operator (Theorem 1). Note that this Lemma is an adapted version of a property defined in (Hansson, 1999).

Lemma 1. $K_A^{\top -\bot} \alpha = K_A^{\top -\bot} \beta$ if and only if for all subsets $K'$ of $K$: $\alpha \in \text{Bel}(K')$ if and only if $\beta \in \text{Bel}(K')$.

Proof

We will use reducito by absurdum.

($\Rightarrow$) Suppose that there is some subset $B$ of $K$ such that $\alpha \in \text{Bel}(B)$ and $\beta \not\in \text{Bel}(B)$. By compactness, there is some subset $B'$ of $K$ such that $\alpha \in \text{Bel}(B')$. Then, there is some element $B''$ of $K^{\top -\bot} \alpha$ such that $B'' \subseteq B'$. Since $B'' \subseteq B$ and $\beta \not\in \text{Bel}(B)$, we have $\beta \not\in \text{Bel}(B'')$, so that $B'' \not\in K^{\top -\bot} \alpha$. Then $B'' \in K^{\top -\bot} \alpha$ and $B'' \not\in K^{\top -\bot} \beta$ contrary to $K^{\top -\bot} \alpha = K^{\top -\bot} \beta$.

($\Leftarrow$) Suppose that $K^{\top -\bot} \alpha \neq K^{\top -\bot} \beta$. We may assume that there is some $X \in K^{\top -\bot} \alpha$ such that $X \not\in K^{\top -\bot} \beta$. There are two cases:

- $\beta \not\in \text{Bel}(X)$; then we have $\alpha \in \text{Bel}(X)$ and $\beta \not\in \text{Bel}(X)$, showing that the conditions of the lemma are not satisfied.
- \( \beta \in \text{Bel}(X) \): then it follows from \( X \not\subseteq K^\uparrow \uplus \beta \) that there is some \( X' \) such that \( X' \subset X \) and \( \beta \in \text{Bel}(X') \). It follows from \( X' \subset X \subset K^\uparrow \uplus \alpha \) that \( \alpha \not\in \text{Bel}(X') \). We then have \( \beta \in \text{Bel}(X') \) and \( \alpha \not\in \text{Bel}(X') \), showing that the conditions of the lemma are not satisfied.

**Theorem 1.** Let \( K_{A_i} \in \mathcal{A} \) and let "\(-\sigma_i\)" be a contraction operator. "\(-\sigma_i\)" is an optimal contraction using plausibility for \( K_{A_i} \) if and only if it satisfies CP-1, ..., CP-4, i.e., it satisfies success, inclusion, uniformity and minimal plausibility change.

**Proof**

*Postulates to Construction.* We need to show that if an operator \((-\) satisfies the enumerated postulates, then it is possible to build an operator in the way specified in the theorem \((-\sigma_i\)\). Let "\(\sigma_i\)" be a function such that, for every base \( K_{A_i} \) \((K_{A_i} \in \mathcal{A})\) and for every consistent belief \( \alpha \), it holds that:

[Hypothesis] \( \sigma_i(K_{A_i} \uplus \alpha) = K_{A_i} \setminus K_{A_i} - \alpha. \)

We must show:

- **Part A.**
  1. "\(\sigma_i\)" is a well defined function.
  2. \( \sigma_i(K_{A_i} \uplus \alpha) \subseteq \bigcup(K_{A_i} \uplus \alpha) \).
  3. If \( X \in K_{A_i} \uplus \alpha, X \neq \emptyset \), then \( X \cap \sigma_i(K_{A_i} \uplus \alpha) \neq \emptyset \).
  4. If \( (\beta, A_j) \in \sigma_i(K_{A_i} \uplus \alpha) \) then \( (\beta, A_j) \in X \subseteq K_{A_i} \uplus \alpha \) and for all \( (\delta, A_k) \in X \) it holds that \( A_j \subseteq A_k \).
- **Part B.** "\(-\sigma_i\)" is equal to "\(-\)" that is, \( K_{A_i} - \sigma_i \alpha = K_{A_i} - \alpha \).

**Part A.**

1. "\(\sigma_i\)" is a well defined function.

   Let \( \alpha \) and \( \beta \) two sentences such that \( \sigma_i(K_{A_i} \uplus \alpha) = K_{A_i} \uplus \beta \). We need to show that \( \sigma_i(K_{A_i} \uplus \alpha) = K_{A_i} \uplus \beta \). It follows from \( K_{A_i} \uplus \alpha = K_{A_i} \uplus \beta \), by Lemma 1, for all subset \( K' \) of \( K_{A_i} \), \( \alpha \in \text{Bel}(K') \) if and only if \( \beta \in \text{Bel}(K') \). Since \( \text{Sen}(K_{A_i}) = \text{Sen}(K_{A_i}) \) and \( K_{A_i} \subseteq A_k \), then for all subset \( K'' \) of \( K_{A_i} \), \( \alpha \in \text{Bel}(K'') \) if and only if \( \beta \in \text{Bel}(K'') \). Thus, by uniformity, \( K_{A_i} - \alpha = K_{A_i} - \beta \). Therefore, by the definition of \( \sigma_i \) adopted in the hypothesis, \( \sigma_i(K_{A_i} \uplus \alpha) = \sigma_i(K_{A_i} \uplus \beta) \).

2. \( \sigma_i(K_{A_i} \uplus \alpha) \subseteq \bigcup(K_{A_i} \uplus \alpha) \).

   Let \( (\beta, A_j) \in \sigma_i(K_{A_i} \uplus \alpha) \). By the definition of \( \sigma_i \) adopted in the hypothesis \((\beta, A_j) \in (K_{A_i} \setminus K_{A_i} - \alpha) \). Thus, \( (\beta, A_j) \in K_{A_i} \) and \( (\beta, A_j) \not\in K_{A_i} - \alpha \). It follows by minimal plausibility change that there is some \( K' \subseteq K_{A_i} \) such that \( \alpha \not\in \text{Bel}(K') \) and \( \alpha \in \text{Bel}(K' \cup \{(\beta, A_j)\}) \). By compactness, there is some finite subset \( K'' \) of \( K' \) such that \( \alpha \not\in \text{Bel}(K'' \cup \{(\beta, A_j)\}) \). Since \( \alpha \not\in \text{Bel}(K') \) we have \( \alpha \not\in \text{Bel}(K'') \). It follows from \( \alpha \not\in \text{Bel}(K'') \) and \( \alpha \in \text{Bel}(K'' \cup \{(\beta, A_j)\}) \) that there is some \( \alpha \)-kernel that contains \((\beta, A_j)\). Hence, \((\beta, A_j) \in \bigcup(K_{A_i} \uplus \alpha) \).

3. If \( X \in K_{A_i} \uplus \alpha, X \neq \emptyset \), then \( X \cap \sigma_i(K_{A_i} \uplus \alpha) \neq \emptyset \).

   Let \( \emptyset \neq X \in K_{A_i} \uplus \alpha \), we need to show that \( X \cap \sigma_i(K_{A_i} \uplus \alpha) \neq \emptyset \). We should prove that, there exists \((\beta, A_j) \in X \) such that \((\beta, A_j) \in \sigma_i(K_{A_i} \uplus \alpha) \). By success, \( \alpha \not\in \text{Bel}(K_{A_i} - \alpha) \). Since \( X \neq \emptyset \) then \( \alpha \not\in \text{Bel}(X) \) and \( X \not\subseteq K_{A_i} - \alpha \); i.e., there is some \((\beta, A_j) \) such that \((\beta, A_j) \in X \) and \((\beta, A_j) \not\in K_{A_i} - \alpha \). Since \( X \subseteq K_{A_i} \) it follows that \((\beta, A_j) \in (K_{A_i} \setminus K_{A_i} - \alpha) \); i.e., by the definition of \( \sigma_i \) adopted in the hypothesis \((\beta, A_j) \in \sigma_i(K_{A_i} \uplus \alpha) \). Therefore, \( X \cap \sigma_i(K_{A_i} \uplus \alpha) \neq \emptyset \).
4. If \((\beta, A_j) \in \sigma_1(K^\dagger_{A_i} \vdash \alpha)\) then \((\beta, A_j) \in X \subseteq K^\dagger_{A_i} \vdash \alpha\) and for all \((\delta, A_k) \in X\) it holds that \(A_j \leq A_i \subseteq A_k\).

Suppose that \((\beta, A_j) \in \sigma_1(K^\dagger_{A_i} \vdash \alpha)\). Then, by the definition of \(\sigma_1\) adopted in the hypothesis, \((\beta, A_j) \in (K^\dagger_{A_i} \setminus K_{A_i} \vdash \alpha)\). Thus, \((\beta, A_j) \in K^\dagger_{A_i}\) and \((\beta, A_j) \notin K_{A_i} \vdash \alpha\). It follows by minimal plausibility change that there is some \(K' \subseteq K^\dagger_{A_i}\) such that \(\alpha \notin Bel(K')\), but \(\alpha \in Bel(K' \cup \{(\beta, A_j)\})\) and for all \((\delta, A_k) \in K'\) such that \(\alpha \notin Bel((K' \cup \{(\beta, A_j)\}) \setminus \{(\delta, A_k)\})\) it holds that \(A_j \leq A_k \subseteq A_i\). By compactness, there is some finite subset \(K''\) of \(K'\) such that \(\alpha \in Bel(K'' \cup \{(\beta, A_j)\}).\) Since \(\alpha \notin Bel(K')\) we have \(\alpha \notin Bel(K'')\). It follows from \(\alpha \notin Bel(K'')\) and \(\alpha \in Bel(K'' \cup \{(\beta, A_j)\})\) that there is some \(\alpha\)-kernel \(X\) that contains \((\beta, A_j)\). Then, for all \((\delta, A_k) \in X\), \(\alpha \notin Bel(X \setminus \{(\delta, A_k)\})\). Since \(X \subseteq K'\), it follows that \(A_j \leq A_i \subseteq A_k\).

**Part B.** \(\neg \sigma_1\) is equal to \(\neg\), that is, \(K_{A_i} \vdash \neg \sigma_1 \alpha = K_{A_i} \vdash \neg \alpha\).

Let \(\neg \sigma_1\) a contraction operator defined as \(K_{A_i} \neg \sigma_1 \alpha = K_{A_i} \setminus X\) where \(X = \{ (\omega, A_j) : \omega \in Sen(\sigma_1(K^\dagger_{A_i} \vdash \alpha)) \} \) and \(\omega, A_j \in K_{A_i}\) and \(\sigma_1\) defined as in the hypothesis.

(≥) Let \((\delta, A_j) \in K_{A_i}\). Then it follows by inclusion that \(K_{A_i} \vdash \neg \alpha \subseteq K_{A_i}\) and \((\delta, A_j) \in K_{A_i}\). Thus, it follows from \((\delta, A_j) \in K_{A_i} \setminus \alpha\) and \((\delta, A_j) \in K_{A_i}\) that \((\delta, A_j) \notin (K_{A_i} \setminus \alpha)\). Since \(K^\dagger_{A_i} \subseteq K_{A_i}\), then \((\delta, A_j) \notin (K^\dagger_{A_i} \setminus K_{A_i} \vdash \alpha)\). Thus, by the definition of \(\sigma_1\) adopted in the hypothesis, \((\delta, A_j) \notin \sigma_1(\beta^\dagger_{A_i} \vdash \alpha)\). We have two cases:

- \((\delta, A_j) \in K^\dagger_{A_i}\). Then \(\delta \notin Sen(\sigma_1(\beta^\dagger_{A_i} \vdash \alpha))\).
- \((\delta, A_j) \in K_{A_i}\). Then, if \((\delta, A_k) \in K^\dagger_{A_i}\), it holds that \(A_j \leq A_i \subseteq A_k\). By reductio ad absurdum, suppose that \((\delta, A_k) \in \sigma_1(\beta^\dagger_{A_i} \vdash \alpha)\). Then \((\delta, A_k) \in (K^\dagger_{A_i} \setminus K_{A_i} \vdash \alpha)\) by the definition of \(\sigma_1\) adopted in the hypothesis. Then \((\delta, A_k) \notin K_{A_i} \vdash \alpha\) which is absurd due to what we supposed that \((\delta, A_j) \in K_{A_i} \vdash \neg \alpha\) and \((\delta, A_k) \in K^\dagger_{A_i}\). Therefore, \((\delta, A_k) \notin \sigma_1(\beta^\dagger_{A_i} \vdash \alpha)\) and \(\delta \notin Sen(\sigma_1(\beta^\dagger_{A_i} \vdash \alpha))\).

Hence, it follows from definition that \((\delta, A_j) \in K_{A_i} \neg \sigma_1 \alpha\).

(≤) Let \((\delta, A_j) \in K_{A_i} \neg \sigma_1 \alpha\). By definition \((\delta, A_j) \in K_{A_i} \setminus X\) where \(X = \{ (\omega, A_k) : \omega \in Sen(\sigma_1(\beta^\dagger_{A_i} \vdash \alpha)) \} \) and \((\omega, A_k) \in K_{A_i}\). Then, \((\delta, A_j) \in K_{A_i}\) and \((\delta, A_j) \notin X\). Therefore, \(\delta \notin Sen(\sigma_1(\beta^\dagger_{A_i} \vdash \alpha))\). Thus, by the definition of \(\sigma_1\) adopted in the hypothesis, \(\delta \notin Sen(K^\dagger_{A_i} \setminus (K_{A_i} \vdash \alpha))\). Hence, \(\delta \in Sen(K_{A_i} \vdash \neg \alpha)\) and it must be the case in which \((\delta, A_j) \in K_{A_i} \neg \sigma_1 \alpha\).

• Construction to Postulates. Let \(\neg \sigma_1\) be an optimal contraction using plausibility for \(K_{A_i}\). We need to show that it satisfies the four conditions of the theorem.

(CP1) **Success:** if \(\alpha \notin Cn(\emptyset)\), then \(\alpha \notin Bel(K_{A_i} \neg \sigma_1 \alpha)\).

**Proof.** Suppose to the contrary that \(\alpha \notin Cn(\emptyset)\) and \(\alpha \in Bel(K_{A_i} \neg \sigma_1 \alpha)\). By compactness, there is a finite subset \(K'\) of \(K_{A_i} \neg \sigma_1 \alpha\) such that \(\alpha \in Bel(K')\). There is then an \(\alpha\)-kernel \(K''\) such that \(K'' \subseteq K'\). Since \(K' \subseteq K_{A_i} \neg \sigma_1 \alpha \subseteq K_{A_i}\), \(K''\) is also an \(\alpha\)-kernel of \(K_{A_i}\). We then have \(K'' \subseteq K^\dagger_{A_i} \vdash \alpha\). However, it follows from \(\alpha \notin Cn(\emptyset)\) that \(K'' \neq \emptyset\). By clause (2) of Definition 17, there is some \(\beta \in Sen(K'')\) such that \(\beta \in Sen(\sigma_1(\beta^\dagger_{A_i} \vdash \alpha))\). By Definition 20, \(\beta \notin Sen(K_{A_i} \neg \sigma_1 \alpha)\), contrary to \(\beta \in Sen(K'')\) with \(K'' \subseteq K_{A_i} \neg \sigma_1 \alpha\).

(CP2) **Inclusion:** \(K_{A_i} \neg \sigma_1 \alpha \subseteq K_{A_i}\).

**Proof.** Straightforward by definition.

(CP3) **Uniformity:** If for all \(K' \subseteq K_{A_i}\), \(\alpha \in Bel(K')\) if and only if \(\beta \in Bel(K')\) then \(K_{A_i} \neg \sigma_1 \alpha = K_{A_i} \neg \sigma_1 \beta\).
We must show:

\[ \text{Hypothesis} \]

\[ \text{Theorem 2.} \]

Let \( K_{A_i} \in \mathcal{K} \) and let \( \ast_{\sigma_i} \) be a revision operator. \( \ast_{\sigma_i} \) is a prioritized revision using plausibility for \( K_{A_i} \), if and only if it satisfies RP-1, ..., RP-5, i.e., it satisfies success, inclusion, consistency, uniformity and minimal plausibility change.

**Proof**

- **Postulates to Construction.** We need to show that if an operator (\( \ast \)) satisfies the enumerated postulates, then it is possible to build an operator in the way specified in the theorem (\( \ast_{\sigma_i} \)). Let \( \sigma_i \) be a function such that, for every base \( K_{A_i} \) (\( K_{A_i} \in \mathcal{K} \)) and for every consistent belief \( \alpha \), it holds that:

\[ \text{Hypothesis} \quad \sigma_i(K_{A_i}^{\perp} - \alpha) = K_{A_i}^{\perp} \setminus K_{A_i} \ast (\alpha, A_j). \]

We must show:

- **Part A.**
  1. \( \ast_{\sigma_i} \) is a well defined function.
  2. \( \sigma_i(K_{A_i}^{\perp} - \alpha) \subseteq \bigcup(K_{A_i}^{\perp} - \alpha) \).
  3. If \( X \in K_{A_i}^{\perp} - \alpha \), \( X \neq \emptyset \), then \( X \cap \sigma_i(K_{A_i}^{\perp} - \alpha) \neq \emptyset \).
  4. If \( (\beta, A_j) \in \sigma_i(K_{A_i}^{\perp} - \alpha) \) and \( (\beta, A_j) \in K_{A_i}^{\perp} - \alpha \) and for all \( (\delta, A_k) \in X \) it holds that \( A_j \leq^{A_i}_{C_0} A_k \).

- **Part B.** \( \ast_{\sigma_i} \) is equal to \( \ast \), that is, \( K_{A_i} \ast_{\sigma_i} (\alpha, A_j) = K_{A_i} \ast (\alpha, A_j) \).

**Part A.**

1. \( \ast_{\sigma_i} \) is a well defined function.

   Let \( \neg \alpha \) and \( \neg \beta \) two sentences such that \( K_{A_i}^{\perp} - \alpha = K_{A_i}^{\perp} - \neg \beta \). We need to show that \( \sigma_i(K_{A_i}^{\perp} - \alpha) = \sigma_i(K_{A_i}^{\perp} - \beta) \). It follows from \( K_{A_i}^{\perp} - \alpha = K_{A_i}^{\perp} - \beta \), by Lemma 1, for all subset \( K' \) of \( K_{A_i}^{\perp} - \alpha \in \text{Bel}(K') \) if and only if \( \neg \beta \in \text{Bel}(K') \). That is, for all subset \( K' \) of \( K_{A_i}^{\perp} - \alpha \).
Sen(K’) ⊢ ¬α if and only if Sen(K’) ⊢ ¬β. Then, for all subset K’ of $K^1_{A_i}$, \{α\} ∪ Sen(K’) ⊢ ⊥ if and only if \{β\} ∪ Sen(K’) ⊢ ∥. Since, $K^1_{A_i} \subseteq K_{A_i}$ (I) and $\text{Sen}(K^1_{A_i}) = \text{Sen}(K_{A_i})$, then for all subset K” of $K_{A_i}$, \{α\} ∪ Sen(K”) ⊢ ⊥ if and only if \{β\} ∪ Sen(K”) ⊢ ⊥. Thus, by uniformity, $K_{A_i} \cap (K_{A_i} * \{α, A_j\}) = K_{A_i} \cap (K_{A_i} \{β, A_k\})$. Then, $K_{A_i} \setminus (K_{A_i} * \{α, A_j\}) = K_{A_i} \setminus (K_{A_i} \{β, A_k\})$ (II). Therefore, by definition of σ₁ adopted in the hypothesis, (I) and (II), then $σ_1(K^1_{A_i} \setminus α) = σ_1(K^1_{A_i} \setminus β)$.

2. $σ_1(K^1_{A_i} \setminus α) \subseteq \bigcup(K^1_{A_i} \setminus α)$. Let $(β, A_j) \in σ_1(K^1_{A_i} \setminus α)$. By the definition of σ₁ adopted in the hypothesis $(β, A_j) \in (K^1_{A_i} \setminus K_{A_i} * (α, A_k))$. Thus, $(β, A_j) \in K^1_{A_i} * (α, A_k)$. It follows by minimal plausibility change that $(β, A_j) \in \bigcup(K^1_{A_i} \setminus α)$. Therefore, by definition of σ₁ adopted in the hypothesis $(β, A_j) \in σ_1(K^1_{A_i} \setminus α)$, therefore, $X \cap σ_1(K^1_{A_i} \setminus α) \neq ∅$.

3. If $X \in K^1_{A_i} \setminus α$, $X \neq ∅$, then $X \cap \sigma_1(K^1_{A_i} \setminus α) \neq ∅$.

Let $\emptyset \neq X \in K^1_{A_i} \setminus α$, we need to show that $X \cap \sigma_1(K^1_{A_i} \setminus α) \neq ∅$. We should prove that, there exists $(β, A_j) \in X$ such that $(β, A_j) \in \sigma_1(K^1_{A_i} \setminus α)$. Suppose that α is consistent. Since we have assumed that $K_{A_i}$ is consistent, by consistency, $K_{A_i} * (α, A_k)$ is consistent. Since $X \neq ∅$ and X is inconsistent with α then $X \nsubseteq K_{A_i} * (α, A_k)$ by success. This means that there is some $(β, A_j) \in X$ and $(β, A_j) \notin K_{A_i} * (α, A_k)$. Since $X \subseteq K^1_{A_i}$ it follows that $(β, A_j) \in (K^1_{A_i} \setminus K_{A_i} * (α, A_k))$ i.e., by the definition of σ₁ adopted in the hypothesis $(β, A_j) \in σ_1(K^1_{A_i} \setminus α)$. Therefore, $X \cap \sigma_1(K^1_{A_i} \setminus α) \neq ∅$.

4. If $(β, A_j) \in σ_1(K^1_{A_i} \setminus α)$ then $(β, A_j) \in X \in K^1_{A_i} \setminus α$ and for all $(δ, A_k) \in X$ it holds that $A_j \leq C_{A_i} A_k$.

Suppose that $(β, A_j) \in σ_1(K^1_{A_i} \setminus α)$. Then, by the definition of σ₁ adopted in the hypothesis $(β, A_j) \in (K^1_{A_i} \setminus K_{A_i} * (α, A_k))$. Thus, $(β, A_j) \in K^1_{A_i}$ and $(β, A_j) \notin K_{A_i} * (α, A_k)$. It follows by minimal plausibility change that there is some $K’ \subseteq K_{A_i}$ such that $¬α \notin Bel(K’)$, but $¬α \notin Bel(K’ \setminus K_{A_i} * (β, A_j))$. Then, for all $(δ, A_k) \in K’$ such that $¬α \notin Bel((K’ \setminus \{β, A_j\}) \setminus \{δ, A_k\})$ it holds that $A_j \leq C_{A_i} A_k$. By compactness, there is some finite subset $K’’$ of $K’$ such that $¬α \notin Bel(K’’ \setminus \{δ, A_k\})$. Since $¬α \notin Bel(K’)$ we have $¬α \notin Bel(K’’)$ and $¬α \in Bel(K’’ \setminus \{δ, A_k\})$ that there is some $¬α$-kernel that contains $(β, A_j)$. Then, for all $(δ, A_k) \in X$, $¬α \notin Bel(X \setminus \{δ, A_k\})$. Since $X \subseteq K’$, it follows that $A_j \leq C_{A_i} A_k$.

Part B. “*$σ_1$” is equal to “*$”, that is, $K_{A_i} * σ_1 (α, A_j) = K_{A_i} * (α, A_j)$.

Let “*$σ_1$” a revision operator defined as $K_{A_i} * σ_1 (α, A_j) = (K_{A_i} \setminus X) \cup \{α, A_j\}$ where: $X = \{ω, A_p) : ω ∈ Sen(σ_1(K^1_{A_i} \setminus α)) \}$ and $σ_1$ defined as in the hypothesis.

(2) Let $(δ, A_k) \in K_{A_i} * (α, A_j)$. It follows by inclusion that $K_{A_i} * (α, A_j) \subseteq K_{A_i} \cup \{α, A_j\}$ and $(δ, A_k) \in K_{A_i} \cup \{α, A_j\})$. If $(δ, A_k) = (α, A_j)$ then $(δ, A_k) \in K_{A_i} * σ_1 (α, A_j)$ by definition. Suppose that $(δ, A_k) \neq (α, A_j)$. Since $(δ, A_k) \in K_{A_i} \cup \{α, A_j\}$ then $(δ, A_k) \in K_{A_i}$, thus, it follows from $(δ, A_k) \in K_{A_i} * (α, A_j)$ that $(δ, A_k) \notin (K_{A_i} \setminus K_{A_i} * (α, A_j))$. Since $K_{A_i} \subseteq K_{A_i}$ then $(δ, A_k) \notin (K^1_{A_i} \setminus K_{A_i} \{α, A_j\})$. Therefore, by the definition of σ₁ adopted in the hypothesis, $(δ, A_k) \notin σ_1(K^1_{A_i} \setminus α)$. We have two cases:

- $(δ, A_k) \in K^1_{A_i}$. Then $δ \notin Sen(σ_1(K^1_{A_i} \setminus α))$. 

- $(\delta, A_k) \in K_{\mathcal{A}_i}$. Then, if $(\delta, A_p) \in K_{\mathcal{A}_i}^\uparrow$ it holds that $A_k \leq_{\mathcal{C}_o} A_p$. By reductio ad absurdum, suppose that $(\delta, A_p) \in \sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha)$ then $(\delta, A_p) \in (K_{\mathcal{A}_i}^\uparrow \setminus K_{\mathcal{A}_i}^\uparrow \downarrow \neg \alpha)$ by the definition of $\sigma_1$ adopted in the hypothesis. Then $(\delta, A_p) \notin K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$ which is absurd due to we supposed that $(\delta, A_k) \in K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$ and $(\delta, A_p) \in K_{\mathcal{A}_i}^\uparrow$. Therefore $(\delta, A_p) \notin \sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha)$. Thus $\delta \notin \text{Sen}(\sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha))$.

Therefore, it follows from definition that $(\delta, A_k) \in K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$.

$(\subseteq)$ Let $(\delta, A_k) \in K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$. It follows from definition that $K_{\mathcal{A}_i} \uparrow (\alpha, A_j) \subseteq K_{\mathcal{A}_i} \cup \{(\alpha, A_j)\}$ and $(\delta, A_k) \in K_{\mathcal{A}_i} \cup \{(\alpha, A_j)\}$. Then $(\delta, A_k) \in K_{\mathcal{A}_i}$ or $(\delta, A_k) = (\alpha, A_j)$. We have two cases:

- $(\delta, A_k) = (\alpha, A_j)$. Then, by success, $(\delta, A_k) \in K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$.
- $(\delta, A_k) \in K_{\mathcal{A}_i}$ and $(\delta, A_k) \neq (\alpha, A_j)$. Then, by definition, $\delta \notin \text{Sen}(\sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha))$. By the definition of $\sigma_1$ adopted in the hypothesis, then $\delta \notin \text{Sen}(K_{\mathcal{A}_i}^\uparrow \setminus (K_{\mathcal{A}_i} \uparrow (\alpha, A_j)))$. From $(\delta, A_k) \in K_{\mathcal{A}_i}$ then $\delta \in \text{Sen}(K_{\mathcal{A}_i})$. Since $\text{Sen}(K_{\mathcal{A}_i}) = \text{Sen}(K_{\mathcal{A}_i}^\uparrow)$ then $\delta \in \text{Sen}(K_{\mathcal{A}_i} \uparrow (\alpha, A_j))$. Therefore, it must be the case in which $(\delta, A_k) \in K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$.

• **Construction to Postulates.** Let $\uparrow (\alpha, A_j)$ be an prioritized revision using plausibility for $K_{\mathcal{A}_i}$. We need to show that it satisfies the five conditions of the theorem.

(RP-1) **Success:** $(\alpha, A_j) \in K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$.

**Proof.** Straightforward by definition.

(RP-2) **Inclusion:** $K_{\mathcal{A}_i} \uparrow (\alpha, A_j) \subseteq K_{\mathcal{A}_i} \cup \{(\alpha, A_j)\}$.

**Proof.** Straightforward by definition.

(RP-3) **Consistency:** if $\alpha$ is consistent then $K_{\mathcal{A}_i} \uparrow (\alpha, A_j)$ is consistent.

**Proof.** Straightforward by definition.

(RP-4) **Uniformity:** If for all $K' \subseteq K_{\mathcal{A}_i}$, $\{\alpha\} \cup \text{Sen}(K') \vdash \bot$ if and only if $\{\beta\} \cup \text{Sen}(K') \vdash \bot$ then $K_{\mathcal{A}_i} \cap (K_{\mathcal{A}_i} \uparrow (\alpha, A_j)) = K_{\mathcal{A}_i} \cap (K_{\mathcal{A}_i} \uparrow (\beta, A_k))$.

**Proof.** Suppose that for all subset $K'$ of $K_{\mathcal{A}_i}$, $\{\alpha\} \cup \text{Sen}(K') \vdash \bot$ if and only if $\{\beta\} \cup \text{Sen}(K') \vdash \bot$. Then, it follows that $\text{Sen}(K') \vdash \neg \beta$ if and only if $\text{Sen}(K') \vdash \neg \beta$; i.e., $\neg \beta \in \text{Bel}(K')$ if and only if $\neg \beta \in \text{Bel}(K')$. Hence, by Lemma 1, $K_{\mathcal{A}_i} \uparrow \neg \alpha = K_{\mathcal{A}_i} \uparrow \neg \beta$. Since $\neg \beta$ is a well defined function then $\sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha) = \sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \beta)$. Therefore, by Definition 20, $K_{\mathcal{A}_i} \neg \sigma_1 \neg \alpha = K_{\mathcal{A}_i} \neg \sigma_1 \neg \beta$.

Then, by Definition 21, $K_{\mathcal{A}_i} \cap (K_{\mathcal{A}_i} \uparrow (\alpha, A_j)) = K_{\mathcal{A}_i} \cap (K_{\mathcal{A}_i} \uparrow (\beta, A_k))$.

(RP-5) **Minimal Plausibility Change:** If $(\beta, A_p) \in K_{\mathcal{A}_i}$ and $(\beta, A_p) \notin K_{\mathcal{A}_i} \uparrow (\alpha, A_k)$ then there is $K' \subseteq K_{\mathcal{A}_i}$ where $\neg \alpha \notin \text{Bel}(K')$ but there exists $(\beta, A_j) \in K_{\mathcal{A}_i}$ such that:

- $\neg \alpha \in \text{Bel}(K' \cup \{(\beta, A_j)\})$
- $p = j$ or $A_p \leq_{\mathcal{C}_o} A_j$, and
- for all $(\delta, A_k) \in K'$ such that $\neg \alpha \notin \text{Bel}(K' \cup \{(\beta, A_j)\}) \setminus \{(\delta, A_k)\}$ it holds that $A_j \leq_{\mathcal{A}_i} A_k$.

**Proof.** Suppose $(\beta, A_p) \in K_{\mathcal{A}_i}$ and $(\beta, A_p) \notin K_{\mathcal{A}_i} \uparrow (\alpha, A_k)$. Then, by Definition 21, $(\beta, A_p) \notin (K_{\mathcal{A}_i} \neg \sigma_1 \neg \alpha) + (\alpha, A_k)$. Thus, by Definition 20, $(\beta, A_p) \in \{\omega, A_q\} : \omega \in \text{Sen}(\sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha))$ and $(\omega, A_q) \in K_{\mathcal{A}_i}$). Then $\beta \in \text{Sen}(\sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha))$. By Definition 18 of bottom incision function, $\sigma_1(K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha) \subseteq \bigcup (K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha)$, so that there is some information object $(\beta, A_j)$ such that $(\beta, A_j) \in X \subseteq K_{\mathcal{A}_i}^\uparrow, \downarrow \neg \alpha$. It follows from Definition 7 that $(\beta, A_j) \in K_{\mathcal{A}_i}^\uparrow$. Thus, $p = j$ or $A_p \leq_{\mathcal{C}_o} A_j$. Let $K' \subseteq K_{\mathcal{A}_i}$ such that $X \setminus \{(\beta, A_j)\} \subseteq K'$. We have two cases:
K
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K

Carlos Alchourrón and David Makinson. On the logic of theory change: Safe contraction.

Proof.

(RP-5) Suppose that for all subset

Proof.

By Definition 21,

Consistency

(RP-3) It follows from CP-2 that

Success:

Proof.

(K
¬

K
) is minimal, 

¬
α \not\in Bel(K') but 

¬
α \in Bel(K' \cup \{(\beta, A_j)\}),

for all \(\delta, A_k \in K'\), \n
¬
α \not\in Bel((K' \cup \{(\beta, A_j)\}) \setminus \{(\delta, A_k)\}).

Hence, by Definition 18, 

A_j \leq_{A_\alpha} A_k.

X \setminus \{(\beta, A_j)\} \subset K' and \{(\beta, A_j)\} \not\in K'.

Then, there exists \((\delta, A_k) \in K'\) such that

A_k <_{A_\alpha} A_j

and, by Definition 18, 

(\delta, A_k) \not\in X.

Therefore,

¬
α \not\in Bel((K' \cup \{(\beta, A_j)\}) \setminus \{(\delta, A_k)\})

Proposition 6. If "\(\downarrow\)" satisfies EP-1,...,EP-5 and "\(\neg_{\sigma_i}\)" satisfies CP-1,...,CP-4 then "\(\ast_{\sigma_i}\)" satisfies RP-1,...,RP-5.

Proof

Let "\(\ast_{\sigma_i}\)" be a prioritized revision using plausibility for \(K_{A_i}\), defined as

\[ K_{A_i} \ast_{\sigma_i} (\alpha, A_j) = (K_{A_i} \neg_{\sigma_i} \neg \alpha) + (\alpha, A_j). \]

We need to show that it satisfies RP-1,...,RP-5 from the postulates of expansion using plausibility and from the postulates of optimal contraction using plausibility.

(RP-1) Success: \( (\alpha, A_j) \in K_{A_i} \ast_{\sigma_i} (\alpha, A_j) \).


(RP-2) Inclusion: \( K_{A_i} \ast_{\sigma_i} (\alpha, A_j) \subseteq K_{A_i} \cup \{(\alpha, A_j)\} \).

Proof. It follows from CP-2 that \( K_{A_i} \neg_{\sigma_i} \neg \alpha \subseteq K_{A_i} \). Then, \((K_{A_i} \neg_{\sigma_i} \neg \alpha) \cup \{(\alpha, A_j)\} \subseteq K_{A_i} \cup \{(\alpha, A_j)\} \). Thus, by Definition 16 \((K_{A_i} \neg_{\sigma_i} \neg \alpha) + (\alpha, A_j) \subseteq K_{A_i} \cup \{(\alpha, A_j)\} \). Hence, by Definition 21, \( K_{A_i} \ast_{\sigma_i} (\alpha, A_j) \subseteq K_{A_i} \cup \{(\alpha, A_j)\} \).

(RP-3) Consistency: if \( \alpha \) is consistent then \( K_{A_i} \ast_{\sigma_i} (\alpha, A_j) \) is consistent.

Proof. By Definition 21, \( K_{A_i} \ast_{\sigma_i} (\alpha, A_j) = (K_{A_i} \neg_{\sigma_i} \neg \alpha) + (\alpha, A_j) \). From EP-1 and CP-1 it follows that \( K_{A_i} \ast_{\sigma_i} (\alpha, A_j) \) is consistent.

(RP-4) Uniformity: If for all \( K' \subseteq K_{A_i} \), \{(\alpha) \cup Sen(K') \vdash \bot\) if and only if \( (\beta) \cup Sen(K') \vdash \bot \) then \( K_{A_i} \cap (K_{A_i} \ast_{\sigma_i} (\alpha, A_j)) = K_{A_i} \cap (K_{A_i} \ast_{\sigma_i} (\beta, A_k)) \).

Proof. Suppose that for all subset \( K' \) of \( K_{A_i} \), \( \{(\alpha) \cup Sen(K') \vdash \bot\) if and only if \( (\beta) \cup Sen(K') \vdash \bot \). Then, it follows that \( Sen(K') \vdash \neg \alpha \) if and only if \( Sen(K') \vdash \neg \beta \); i.e., \( \neg \alpha \not\in Bel(K') \) if and only if \( \neg \beta \not\in Bel(K') \). Thus, it follows from CP-3 that \( K_{A_i} \neg_{\sigma_i} \neg \alpha = K_{A_i} \neg_{\sigma_i} \neg \beta \). Hence, by Definition 21, \( K_{A_i} \cap (K_{A_i} \ast_{\sigma_i} (\alpha, A_j)) = K_{A_i} \cap (K_{A_i} \ast_{\sigma_i} (\beta, A_k)) \).

(RP-5) Minimal Plausibility Change: If \((\beta, A_p) \in K_{A_i}\) and \( (\beta, A_p) \not\in K_{A_i} \ast_{\sigma_i} (\alpha, A_k) \) then there is \( K' \subseteq K_{A_i} \) where \( \neg \alpha \not\in Bel(K') \) but there exists \( (\beta, A_j) \in K_{A_i} \) such that:

- \( \neg \alpha \not\in Bel(K' \cup \{(\beta, A_j)\}) \),
- \( p = j \) or \( A_j \leq_{A_\alpha} A_p \), and
- for all \( (\delta, A_k) \in K' \) such that \( \neg \alpha \not\in Bel((K' \cup \{(\beta, A_j)\}) \setminus \{(\delta, A_k)\}) \) it holds that \( A_j \leq_{A_\alpha} A_k \).

Proof. Suppose \((\beta, A_p) \in K_{A_i}\) and \( (\beta, A_p) \not\in K_{A_i} \ast_{\sigma_i} (\alpha, A_k) \). Then, by Definition 21, \( (\beta, A_p) \not\in K_{A_i} \neg_{\sigma_i} \neg \alpha \). Thus, it follows by CP-4 that all conditions of the postulate are satisfied.

References


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